Temperature distribution in a uniformly moving medium

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Abstract. We apply several physical ideas to determine the steady temperature distribution in a medium moving with uniform velocity between two infinite parallel plates. We compute it in the coordinate frame moving with the medium by integration over the "past" to account for the influence of an infinite set of instantaneous point sources of heat in past moments as seen by an observer moving with the medium. The boundary heat flux is simulated by appropriately distributed point heat sources on the inner side of an adiabatically insulating boundary. We make an extensive use of Green functions with an emphasis of their physical meaning. The methodology used in this paper is of great pedagogical value as it offers an opportunity for students to see the connection between powerful mathematical techniques and their physical interpretation in an intuitively clear physical problem. We suggest several problems and a challenging project that can be easily incorporated in undergraduate or graduate courses.

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1. Introduction

In this paper we discuss some physical ideas that can be used to compute the temperature distribution in a medium moving with velocity that is uniform in space. Then we apply these ideas to compute the temperature in a medium moving with uniform velocity in positive x direction between two parallel planes; we will refer to this situation as a *channel flow*. We assume that the temperature at one of the walls is constant in space and time, and without loss of generality take its value to be zero. On the other wall we assume that the heat flux is given. Such problems are common in practice, i.e., cooling of a device with running water flowing in a channel.

Our methods apply also to other geometries, e.g., to the uniform flow in a cylindrical pipe. A classic problem in heat transfer is the so-called *Graetz problem* [1] (see also [2, 3, 4]) concerned with the heating of cold fluid entering a channel or a cylindrical pipe with a circular cross-section with heated walls (where either the temperature of the walls or the heat flux through them is controlled). The Graetz problem has been considered in monographs [5], and is a standard topic in books on heat transfer (including numerical aspects [6]). Even a perfunctory search in a journal like *International Journal of Heat and Mass Transfer* gives hundreds of related articles.

Since heat and diffusion are governed by the same equation, our methods can be applied to the propagation of some substance in a windy atmosphere – not only propagation of smoke (see the early paper [7]), but also the propagation of seeds and pollen [8]. Another important application is the problem of removing of a substance from a moving fluid by an adsorbing wall – a case applicable to hemodialysis [9]. It is difficult to overestimate the practical importance of heat and mass transfer across channel walls in biological systems (in blood vessels, kidneys, lungs), in heaters, nuclear reactors, desalination units, etc.

Clearly, the assumption of spatial uniformity of the velocity of the medium is quite restrictive. Solving the more realistic problem with nonuniform velocity, however, is much more difficult, and an exact solution is not known. After Graetz, the problem was studied by Boussinesq, Nusselt, and others; early works related more closely to the present paper are [10] (see [2] for other old references). The problem has been studied for flows in a channel [11, 12] or in an infinite cylinder [12, 13] with different boundary conditions. Their authors obtained approximate solutions using various assumptions, e.g., neglecting the heat conduction in the direction of the flow in comparison with the heat advected by the flow.

If one is concerned only with the large-scale features in the flow, a uniform flow can be used as a very simple model of a turbulent flow. For example, heat propagation in a very turbulent flow can be thought of as a propagation of heat in a laminar flow approximating the turbulent flow; naturally, in such an approximation, the effective coefficient of thermal conductivity will be enhanced by the turbulent mixing. Such ideas were used in [7], before the development of the modern theory of turbulence.

If the moving medium is a solid body, then our methods produce an exact solution. Problems related to heat propagation in solid objects occur in welding [14] or in the heating of a moving solid body caused by friction with a stationary body, like in grinding [15].

The idealized problem of uniform flow considered in this paper brings together several beautiful physical ideas and relates them with some mathematical techniques. An additional advantage of studying heat propagation is that the behavior of its solutions is intuitively clear. Although we solve one particular problem, the ideas we use are applicable to a wide range of problems. In Appendix A we give a list of problems that can be used in courses of physics, partial differential equations, etc.

Here are some of the physical ideas we employ. To solve a boundary value problem (BVP) with a given flux (i.e., normal derivative) at the boundary, we instead solve the problem with an adiabatically insulated boundary, and then introduce fictitious heat sources at the inner side of the insulated boundary in such a way that their flux is equal to the flux we want to simulate. Another idea is to find the temperature in a uniformly moving medium by using a coordinate frame that is moving with the medium. In this new frame the heat sources (assumed to be at rest in the lab frame) will appear to be moving. Within this approach, to compute the temperature at some fixed point $\mathbf{x}$ in the channel, we consider the particular particle of the medium that is at $\mathbf{x}$ at time 0, and follow this particle throughout its whole past, i.e., from $t = -\infty$ to $t = 0$. From point of view of this moving particle of the medium, the heat source acts as a source of heat that is moving against the particle, or, equivalently, as infinitely many instantaneous point sources of heat, each emitting heat at an appropriately chosen past moment. This gives the temperature of this particle of the medium as an integral over the "past".

We make extensive use of Green functions and their physical interpretation. This powerful technique is widely used in problems on heat propagation and diffusion [16]. Green functions are covered in most textbooks of mathematics for physicists and engineers [17], and are the main topic in [18].

2. Green function for uniform channel flow

2.1. Setting up the BVP and non-dimensionalizing

We first derive the equation of heat propagation in a moving medium. Let $\mathbf{v}(\mathbf{x}, t)$ be the velocity of the medium at the point $\mathbf{x}$ at time t. The density of the heat flux in the medium, $\mathbf{j}(\mathbf{x}, t)$, is due not only to the temperature gradient, but also to the heat that the medium carries due to its motion. Using the Fourier law of heat conduction and taking into account the motion of the medium, we obtain

$$\mathbf{j} = -k\nabla T + c\rho T\mathbf{v} \ , \tag{1}$$

where $k > 0$ is the *thermal conductivity* of the medium (unit: $\mathrm{kg\,m\,s^{-3}\,K^{-1}}$), c is the specific heat (unit: $\mathrm{m^2\,s^{-2}\,K^{-1}}$), and ρ is the mass density of the medium (unit: $\mathrm{kg\,m^{-3}}$). Assume that there are sources of heat distributed so that the volume density of their power is $\Psi(\mathbf{x}, t)$ (unit: $\mathrm{W\,m^{-3}} = \mathrm{kg\,m^{-1}\,s^{-3}}$). From (1), one can obtain the equation governing the heat propagation in a moving medium (see Problem 1 in Appendix A):

$$c\rho\partial_t T = \nabla \cdot (k\nabla T) - \nabla \cdot (c\rho T\mathbf{v}) + \Psi \ . \tag{2}$$

We assume that the temperature does not vary much, so that the physical properties of the fluid – in this case k, c and ρ – can be treated as constants. Throughout the paper we assume that $\mathbf{v} = v\mathbf{i}$, where v is constant; hereafter, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the coordinate axes.

The boundary conditions (BCs) for (2) we will study are the following. Let the medium be moving in the domain between the "lower" and "upper" planes, $z = 0$ and $z = \ell$ respectively (ℓ is a positive constant). Let the upper plane be kept at zero temperature, while the surface density σ of the heat flux entering the channel through the lower boundary be a given function of x, y and t. By Fourier law, $\sigma(x, y, t) = \mathbf{k} \cdot (-k\nabla T|_{z=0}) = -k\partial_z T|_{z=0}$. In the case of channel flow we assume that the medium is heated only at the boundary, i.e., that $\Psi \equiv 0$ in (2) (although the idea described below can be applied also to the case of arbitrary Ψ). We obtain the following BVP for the steady temperature distribution $T(\mathbf{x})$ in the channel:

$$\begin{aligned} &\frac{k}{c\rho}\,\Delta T(\mathbf{x}) - v\partial_x T(\mathbf{x}) = 0\,, \qquad && (x, y) \in \mathbb{R}^2\,,\ z \in (0, \ell)\,, \\ &\partial_z T|_{z=0} = -\tfrac{1}{k}\sigma(x, y)\,, \qquad && T|_{z=\ell} = 0\,. \end{aligned} \tag{3}$$

To simplify the exposition, we will mainly be interested in heat flux of the form $\sigma(x, y) = \sigma_c H(x)$, where $\sigma_c > 0$ is a constant, and H is the Heaviside function (the "unit step function").

We non-dimensionalize the problem by introducing the following units for length, time, and temperature: $x_\mathrm{u} := \ell$, $t_\mathrm{u} := c\rho\ell^2/k$, $T_\mathrm{u} := \sigma_c\ell/k$. In these units, the BVP (3) becomes

$$\Delta T(\mathbf{x}) - v\partial_x T(\mathbf{x}) = 0\,, \qquad \partial_z T|_{z=0} = -H(x)\,, \quad T|_{z=1} = 0\,, \tag{4}$$

where the ranges of the non-dimensional variables are $(x, y) \in \mathbb{R}^2$, $z \in (0, 1)$.

2.2. Method of images, eigenfunction expansion

Recall that a Dirichlet boundary condition (BC) for a BVP specifies the value that the solution must take on the boundary, while a Neumann BC specifies the value that the normal derivative of the solution must take on the boundary. Any solution satisfying a Dirichlet (resp. Neumann) BC may be redefined to satisfy the zero Dirichlet (resp. Neumann) BC, so we concern ourselves only with the latter. We first compute the "Neumann-Dirichlet" Green function $G_{\mathrm{ND}}(\mathbf{x},t;\boldsymbol{\xi},\tau)$ of the (time-dependent) heat equation in a stationary medium with a zero Neumann BC on the lower and a zero Dirichlet BC on the upper plane, i.e., the solution of the BVP

$$\begin{aligned}&\partial_t G_{\mathrm{ND}} - \Delta_{\mathbf{x}} G_{\mathrm{ND}} = \delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t-\tau)\,,\\ &\partial_z G_{\mathrm{ND}}|_{z=0} = 0\,, \quad G_{\mathrm{ND}}|_{z=1} = 0\,, \quad G_{\mathrm{ND}}|_{t<\tau} = 0\,,\end{aligned} \tag{5}$$

where $\mathbf{x} = (x,y,z)$ and $\boldsymbol{\xi} = (\xi,\eta,\zeta)$ with $z \in (0,1)$, $\zeta \in (0,1)$, $t \in \mathbb{R}$, $\tau \in \mathbb{R}$.

To find G_{ND}, one can use the classical *method of images*. Namely, consider the BVP (5) in the whole space $\mathbb{R}^3$, and place fictitious instantaneous point sources and sinks of heat at appropriate positions (outside the slab $0 \le z \le 1$) to ensure that the temperature distribution generated by them satisfies the BCs from (5). For $\boldsymbol{\xi} = (\xi,\eta,\zeta)$ with $\zeta \in [0,1]$, let $\boldsymbol{\xi}^{\pm}_{2n} := (\xi,\eta,2n\pm\zeta)$. Consider instantaneous point sources of heat placed at the points $\boldsymbol{\xi}^{\pm}_{2n}$ for even n, and instantaneous point sinks of heat placed at the points $\boldsymbol{\xi}^{\pm}_{2n}$ for odd n. Let each instantaneous source/sink emit/absorb a unit amount of heat at time τ. Then the temperature at the point $\mathbf{x}$ at time t is

$$G_{\mathrm{ND}}(\mathbf{x},t;\boldsymbol{\xi},\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \left[G_0^{(3)}(\mathbf{x},t;\boldsymbol{\xi}^{+}_{2n},\tau) + G_0^{(3)}(\mathbf{x},t;\boldsymbol{\xi}^{-}_{2n},\tau)\right]\,, \tag{6}$$

where $G_0^{(3)}$ is the Green function (B.1) (see Appendix B) for the heat equation in $\mathbb{R}^3$. Elementary physical reasoning shows that this function solves the BVP (5) (see Problem 2 in Appendix A).

Alternatively, one can easily show (see, e.g., Problem 3 in Appendix A) that $G_{\mathrm{ND}}(\mathbf{x},t;\boldsymbol{\xi},\tau)$ is the product of the "whole-plane" Green function $G_0^{(2)}(x,y,t;\xi,\eta,\tau)$ (B.1) in the variables (x,y), and the "Neumann-Dirichlet" Green function $G_{\mathrm{ND}}^{(1)}(z,t;\zeta,\tau)$ (B.2) (see Appendix B) on the interval $[0,1]$ in z direction. Then the solution of the BVP (5) is

$$\begin{aligned}G_{\mathrm{ND}}(\mathbf{x},t;\boldsymbol{\xi},\tau) &= G_0^{(2)}(x,y,t;\xi,\eta,\tau)\,G_{\mathrm{ND}}^{(1)}(z,t;\zeta,\tau)\\ &= \frac{H(t-\tau)\,\mathrm{e}^{-\frac{(x-\xi)^2+(y-\eta)^2}{4(t-\tau)}}}{4\pi(t-\tau)} \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{(2n+1)^2\pi^2}{4}(t-\tau)}\,\mathrm{C}_n(z)\,\mathrm{C}_n(\zeta)\,.\end{aligned} \tag{7}$$

The equivalence of these two definitions yields a particular case of the *Poisson summation formula* (see Appendix A).

2.3. Moving with the medium

Now we will find the Green function $G_v(\mathbf{x};\boldsymbol{\xi})$ that solves the BVP

$$\Delta G_v - v\partial_x G_v = \delta(\mathbf{x}-\boldsymbol{\xi})\;, \quad \partial_z G_v|_{z=0} = 0\,, \quad G_v|_{z=1} = 0\,, \tag{8}$$

i.e., the steady temperature distribution in the moving medium due to a point source of unit power placed at $\boldsymbol{\xi}$. Note how this differs from the Green function G_{ND}; instead

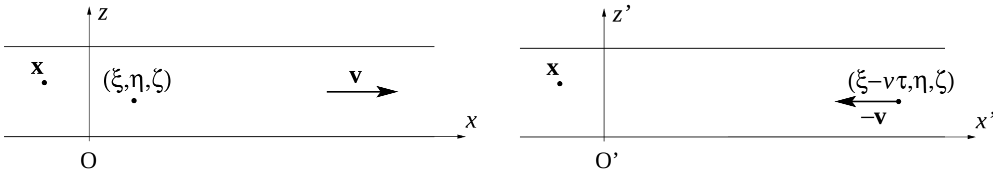

Figure 1. *Left:* The point $\mathbf{x}$ and the source $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ in laboratory frame K. *Right:* The point $\mathbf{x}$ and the source $\boldsymbol{\xi} - \mathbf{v}\tau = (\xi - v\tau, \eta, \zeta)$ (moving with velocity $-\mathbf{v}$) in the frame K' moving with the medium.

of a point source "flash" of heat, G_v represents a point source constantly emitting heat and traveling at speed v, with the medium. Let K be the laboratory coordinate frame (in which the medium is moving with velocity $\mathbf{v}$), and K' be the coordinate frame moving together with the medium, and such that K and K' coincide at time 0. To find $G_v(\mathbf{x}; \boldsymbol{\xi})$ (where $\mathbf{x}$ and $\boldsymbol{\xi}$ are points stationary with respect to K), we think from the point of view of an observer in K' moving together with the medium in such a way that at time 0 the observer's coordinate is $\mathbf{x}$. For this observer (i.e., in K'), the point source is moving with velocity $-\mathbf{v}$, so that at time $\tau < 0$ the coordinate of the source is $\boldsymbol{\xi} - \mathbf{v}\tau$, and this source is "on" for times $\tau \in (-\infty, 0]$ – see Figure 1. Therefore, to obtain $G_v(\mathbf{x}; \boldsymbol{\xi})$, we can integrate $G_{\rm ND}(\mathbf{x}, 0; \boldsymbol{\xi} - \mathbf{v}\tau, \tau)$ over τ from $-\infty$ to 0:

$$
\begin{aligned}
G_v(\mathbf{x}; \boldsymbol{\xi}) &= \int_{-\infty}^{0} G_{\rm ND}(\mathbf{x}, 0; \boldsymbol{\xi} - \mathbf{v}\tau, \tau)\,\mathrm{d}\tau \\
&= -\frac{1}{4\pi} \sum_{n=0}^{\infty} \mathrm{C}_n(z)\,\mathrm{C}_n(\zeta) \int_{-\infty}^{0} \frac{\mathrm{d}\tau}{\tau}\, \mathrm{e}^{\frac{(x-\xi+v\tau)^2+(y-\eta)^2}{4\tau} + \frac{(2n+1)^2\pi^2}{4}\tau} \\
&= \frac{\mathrm{e}^{\frac{v(x-\xi)}{2}}}{2\pi} \sum_{n=0}^{\infty} K_0\!\left(\frac{\sqrt{v^2+(2n+1)^2\pi^2}}{2} \sqrt{(x-\xi)^2 + (y-\eta)^2} \right) \mathrm{C}_n(z)\,\mathrm{C}_n(\zeta)\,. \qquad (9)
\end{aligned}
$$

where we used the substitution $\tau = -\mathrm{e}^{w} \sqrt{\frac{(x-\xi)^2+(y-\eta)^2}{v^2+(2n+1)^2\pi^2}}$ and (B.3) from Appendix B.

3. Simulating boundary flux with sources at an insulating boundary

3.1. The idea and a test case

If a point source of heat is infinitesimally close to an adiabatically insulating boundary, then all the heat from the source will go into the spatial domain in which we solve the BVP. Therefore, flux through the boundary is equivalent to point sources of heat distributed over an adiabatically insulated boundary, hence the solutions of the BVPs

$$\Delta T(\mathbf{x}) - v\partial_x T(\mathbf{x}) = 0\,, \qquad \partial_z T|_{z=0} = -\delta(x-\xi)\delta(y-\eta)\,, \quad T|_{z=1} = 0 \qquad (10)$$

and

$$\Delta T(\mathbf{x}) - v\partial_x T(\mathbf{x}) = \delta(x-\xi)\delta(y-\eta)\delta(z)\,, \qquad \partial_z T|_{z=0} = 0\,, \quad T|_{z=1} = 0 \qquad (11)$$

are the same. The minus sign in front of the delta functions in (10) is due to the minus sign in the Fourier law (cf. the Neumann BC in (3)). Since (11) is a particular case of (8), the solution of the BVP (10) is $T(\mathbf{x}) = G_v(\mathbf{x}; \xi, \eta, 0)$. If the heat flux entering the channel through the lower boundary has area density $\sigma(x, y)$, then the temperature in the medium due to this flux is

$$T(\mathbf{x}) = \int_{\mathbb{R}^2} G_v(\mathbf{x}; \xi, \eta, 0)\,\sigma(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \qquad (12)$$

As a test for the expression (9) and the idea embodied in (12), let us compute the total flux $\Phi(z)$ of a unit-power point source at the origin, $\boldsymbol{\xi} = \mathbf{0}$, through a horizontal plane at height z (where $z \in (0,1)$). In this case, the temperature of the moving medium is $T(\mathbf{x}) = G_v(\mathbf{x};\mathbf{0})$. Since all the heat from the source leaves the slab $0 < z < 1$ through the upper ("cold") plane, $\Phi(z)$ must be 1 for any position of the plane, and for any speed v of the medium. Setting $\gamma_{v,n} = \sqrt{v^2 + (2n+1)^2\pi^2}$, we obtain

$$\begin{aligned}
\Phi(z) &= \int_{\mathbb{R}^2} \mathrm{d}x\,\mathrm{d}y \left(-\frac{\partial T}{\partial z}\right) = -\int_0^\infty \mathrm{d}r\, r \int_0^{2\pi} \mathrm{d}\theta \frac{\partial}{\partial z}\left(\frac{\mathrm{e}^{\frac{vr\cos\theta}{2}}}{2\pi}\sum_{n=0}^{\infty} K_0\big(\tfrac{\gamma_{v,n}}{2}r\big)\,\mathrm{C}_n(z)\sqrt{2}\right) \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty}(2n+1)\pi\sin\frac{(2n+1)\pi z}{2}\int_0^\infty \mathrm{d}r\, r\, K_0\big(\tfrac{\gamma_{v,n}}{2}r\big)\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{\frac{vr\cos\theta}{2}} \\
&= \sum_{n=0}^{\infty}(2n+1)\pi\sin\frac{(2n+1)\pi z}{2}\int_0^\infty \mathrm{d}r\, r\, K_0\big(\tfrac{\gamma_{v,n}}{2}r\big)\, I_0\left(\tfrac{v}{2}r\right) \\
&= \sum_{n=0}^{\infty}\frac{4}{(2n+1)\pi}\sin\frac{(2n+1)\pi z}{2} = 1\,,
\end{aligned}$$

where we used (B.4) and (B.5) from Appendix B, and the Fourier expansion of the periodic function $f(z)$ of period 2 defined by $f(z) = \mathrm{sgn}(z)$ for $-1 < z < 1$.

3.2. Temperature distribution for a constant flux through the lower half-plane

First we find the temperature distribution $\widetilde{T}(\mathbf{x};\xi)$ due to an infinite line of flux perpendicular to the flow of the medium (with abscissa equal to ξ). This temperature is given by the integral (12) with $\sigma(\xi',\eta') = \delta(\xi'-\xi)$:

$$\begin{aligned}
\widetilde{T}(\mathbf{x};\xi) &= \int_{-\infty}^{\infty}\mathrm{d}\eta\, G_v(\mathbf{x};\xi,\eta,0) = \int_{-\infty}^{0}\mathrm{d}\tau\int_{-\infty}^{\infty}\mathrm{d}\eta\, G_0^{(2)}(x,y,0;\xi-v\tau,\eta,\tau)\, G_{\mathrm{ND}}^{(1)}(z,0;0,\tau) \\
&= \int_{-\infty}^{0}\mathrm{d}\tau\, G_0^{(1)}(x,0;\xi-v\tau,\tau)\, G_{\mathrm{ND}}^{(1)}(z,0;0,\tau) \\
&= \frac{1}{\sqrt{4\pi}}\sum_{n=0}^{\infty}\mathrm{C}_n(z)\sqrt{2}\int_{-\infty}^{0}\frac{\mathrm{d}\tau}{\sqrt{-\tau}}\,\mathrm{e}^{\frac{(x-\xi+v\tau)^2}{4\tau}+\frac{(2n+1)^2\pi^2}{4}\tau} \\
&= \frac{1}{\sqrt{4\pi}}\sum_{n=0}^{\infty}\mathrm{C}_n(z)\,2^{3/2}\left(\frac{|x-\xi|}{\gamma_{v,n}}\right)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}v(x-\xi)}\int_0^\infty \mathrm{d}b\,\mathrm{e}^{-\frac{1}{2}\gamma_{v,n}|x-\xi|\cosh b}\cosh\tfrac{b}{2} \\
&= \frac{1}{\sqrt{4\pi}}\sum_{n=0}^{\infty}\mathrm{C}_n(z)\,2^{3/2}\left(\frac{|x-\xi|}{\gamma_{v,n}}\right)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}v(x-\xi)}K_{\frac{1}{2}}\big(\tfrac{1}{2}\gamma_{v,n}|x-\xi|\big) \\
&= \sum_{n=0}^{\infty}\frac{\sqrt{2}\mathrm{C}_n(z)}{\gamma_{v,n}}\mathrm{e}^{\frac{1}{2}[v(x-\xi)-\gamma_{v,n}|x-\xi|]}\,, \qquad (13)
\end{aligned}$$

where we have used (B.3) and $K_{\frac{1}{2}}(a) = \sqrt{\frac{\pi}{2a}}\mathrm{e}^{-a}$.

Now we compute the temperature distribution for a constant flux of heat through

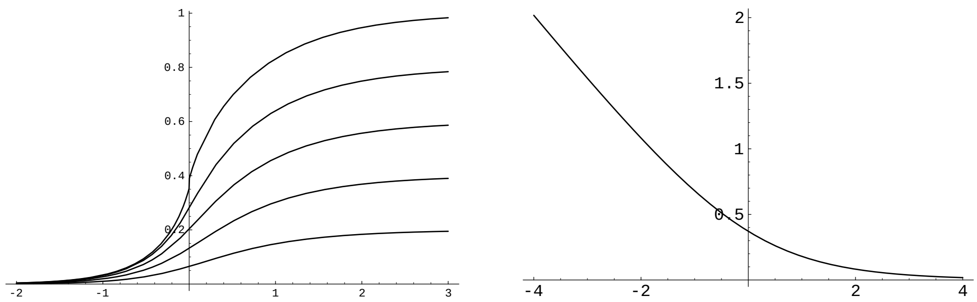

Figure 2. *Left:* Plots of $T(x, y, z)$ (14) vs. x for $z = 0, 0.2, 0.4, 0.6, 0.8, 1$. *Right:* Plot of $\Phi_{\rm back}$ given by (15) as a function of v.

the right half of the lower boundary, i.e., the solution of the BVP (4):

$$T(\mathbf{x}) = \int_0^\infty \mathrm{d}\xi\, \widetilde{T}(\mathbf{x};\xi) = \sum_{n=0}^{\infty} \frac{\sqrt{2}\mathrm{C}_n(z)}{\gamma_{v,n}} \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{\frac{1}{2}[v(x-\xi)-\gamma_{v,n}|x-\xi|]}$$

$$= (1-z)H(x) - \frac{2^{3/2}}{\pi^2}\,\mathrm{e}^{\frac{1}{2}vx}\sum_{n=0}^{\infty}\frac{\mathrm{C}_n(z)}{(2n+1)^2}\left(\frac{v}{\gamma_{v,n}} + \mathrm{sgn}(x)\right)\mathrm{e}^{-\frac{1}{2}\gamma_{v,n}|x|}\,. \qquad (14)$$

Note that this expression behaves as expected for very large $|x|$: when $x \to -\infty$, the temperature goes to zero, while for $x \to \infty$, $T(\mathbf{x}) \approx 1 - z$, as it should. Also, the function (14) is continuous although the Heaviside function and the sgn function are not. The temperature distribution (14) for $v = 1$ is plotted in Figure 2.

An interesting quantity for practical applications is the amount of heat that goes "back", i.e., the total flux through the left ($x < 0$) half of the upper plane per unit length in y direction:

$$\Phi_{\rm back}(v) = -\int_{-\infty}^{0} \frac{\partial T}{\partial z}\bigg|_{z=1} \mathrm{d}x = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{\gamma_{v,n}(v+\gamma_{v,n})^2}\,. \qquad (15)$$

Expanding this expression in Taylor series in v, we obtain

$$\Phi_{\rm back}(v) = \frac{4\beta(2)}{\pi^2} - \frac{8\beta(3)}{\pi^3}\,v - 4\sum_{j=1}^{\infty} v^{2j}\,\frac{(-1)^j(4j^2-1)\beta(2j+2)}{\pi^{2j+2}\,(2j)!}\prod_{k=0}^{j-2}(2k+1)^2\,,$$

where $\beta(k) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k}$ is the Dirichlet beta function (also known as Catalan beta function) [19]; in particular, $\beta(2)$ is equal to the Catalan's constant, $0.915965594\ldots$, and $\beta(3) = \frac{\pi^3}{32}$. For slowly moving medium,

$$\Phi_{\rm back}(v) \approx 0.371227 - \frac{1}{4}\,v + 0.060915\,v^2 - 0.002597\,v^4 + 0.000184\,v^6 + \cdots\,.$$

In the limiting case of large positive v, $\Phi_{\rm back} \to 0$, since the medium's flow keeps the heat flux in the right half of the channel. For large negative v, clearly $\Phi_{\rm back}$ should increase linearly with v since the flow of the medium overpowers the propagation of heat due to conduction. The graph of $\Phi_{\rm back}(v)$ is shown in Figure 2.

The techniques used in this paper are an illustration that Green functions provide a powerful, as well as beautiful and physically meaningful, method for analyzing differential equations. Used in conjunction with intuitive physical reasoning, it is an invaluable tool for any physicist and applied mathematician.

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Appendix A. Suggested problems for students

Problem 1. Derive Equation (2) governing heat propagation in a moving medium.

Hint: Consider an arbitrary bounded domain $\mathcal{D} \subset \mathbb{R}^3$ (not changing with time), apply to it the conservation of heat energy over an infinitesimal time interval δt, and use Gauss law.

Problem 2. Give physical arguments that show that the function $G_{\mathrm{ND}}(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ defined by (6) is a solution of the BVP (5).

Hint: A picture (with all sources and sinks) is worth a thousand words.

Problem 3. Explain physically why for $t > \tau$ the solution of the BVP (5) also solves the initial-boundary value problem

$$\partial_t G - \Delta_{\mathbf{x}} G = 0\,, \quad \partial_z G|_{z=0} = 0\,, \quad G|_{z=0} = 0\,, \quad G|_{t=\tau} = \delta(\mathbf{x} - \boldsymbol{\xi})$$

(in the same spatial domain as in (5)). Can this be used to show that the Green function G_{ND} can be represented as a product as claimed before (7)?

Problem 4. Equating (6) and (7), we obtain an identity which is a particular case of the *Poisson summation formula*,

$$\sum_{n=-\infty}^{\infty} \phi(2\pi n) = \sum_{n=-\infty}^{\infty} \hat{\phi}(n)\,, \qquad \text{where} \quad \hat{\phi}(n) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x) \mathrm{e}^{\mathrm{i}nx}\, \mathrm{d}x\,,$$

for an appropriate choice of a function ϕ. For which function ϕ?

Problem 5. Using both the method of images and the eigenfunction expansion, write down the solution of the Dirichlet-Dirichlet BVP

$$\partial_t G_{\mathrm{DD}} - \Delta_{\mathbf{x}} G_{\mathrm{DD}} = \delta(\mathbf{x} - \boldsymbol{\xi})\, \delta(t - \tau)\,, \quad G_{\mathrm{DD}}|_{z=0} = G_{\mathrm{DD}}|_{z=1} = 0\,, \quad G_{\mathrm{DD}}|_{t<\tau} = 0\,,$$

for $(x, y) \in \mathbb{R}^2$, $z \in (0, 1)$, $t \in \mathbb{R}$, $(\xi, \eta) \in \mathbb{R}^2$, $\zeta \in (0, 1)$, $\tau \in \mathbb{R}$.

Problem 6. Show that a change of the variables from (x, t) to $(x' = x - vt, t' = t)$, and $T'(x', t') := T(x(x', t'), t(x', t'))$, transforms $\partial_t T + v \partial_x T = \partial_{xx} T$ into $\partial_{t'} T' = \partial_{x'x'} T'$. Use this to find the Green function of the equation for $T(x, t)$ on $\mathbb{R}$; the Green function of the 1-dimensional heat equation on $\mathbb{R}$ is given by (B.1).

Problem 7. Show that the function $T(\mathbf{x})$ given by (14) is continuous.

Problem 8. Use (14) to find the steady temperature distribution in a channel flow with zero temperature on the upper wall and zero flux on the lower wall except for the infinite strip $\{x \in [0, L]\,, y \in (-\infty, \infty)\}$, where the flux is constant.

Problem 9. Compute the heat flux $\Phi_{\text{to the left}}(v)$ (per unit length in y direction) of the temperature distribution $T(\mathbf{x})$ (14) to the left through the strip $x = 0$. You should obtain that $\Phi_{\text{to the left}}(v) = \Phi_{\text{back}}(v)$ (15). Is this just a coincidence?

Problem 10. Generalize the methods used in this paper to the case of sources of heat whose power changes with time. Can they also be used if the speed of the medium changes with time (but is still uniform in space)?

Project. Follow the ideas developed in this paper to find the temperature distribution in a medium moving uniformly in an infinite cylindrical pipe with circular cross-section in the following two cases:

(a) if the temperature at the wall is zero, and there are stationary point sources of heat located along the axis, for $z > 0$ (in cylindrical coordinates);

(b) if the walls are adiabatically insulated, except at a circular ring (at $z = 0$) which emits a unit amount of heat every second, and all this heat enters the moving fluid (uniformly in the angular coordinate).

In both cases, compute the total flux going "back" as a function of the velocity of the fluid, similarly to the fluxes found in (15) and in Problem 9.

Appendix B. Collection of formulae

(i) Green function $G_0^{(n)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$ of the heat equation on $\mathbb{R}^n$ ($\mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^n$, $t, \tau \in \mathbb{R}$):

$$\partial_t G_0^{(n)} - \Delta_{\mathbf{x}} G_0^{(n)} = \delta(\mathbf{x} - \boldsymbol{\xi})\, \delta(t - \tau)\,, \qquad \lim_{|\mathbf{x}| \to \infty} G_0^{(n)} = 0\,, \qquad G_0^{(n)}\big|_{t<\tau} = 0\ :$$

$$G_0^{(n)}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \frac{H(t-\tau)}{[4\pi(t-\tau)]^{n/2}}\, \mathrm{e}^{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{4(t-\tau)}}\,. \tag{B.1}$$

(ii) Green function $G_{\mathrm{ND}}^{(1)}(z, t; \zeta, \tau)$ of the 1-dimensional "Neumann-Dirichlet" BVP for the 1-dimensional heat equation on $[0, 1]$ ($z \in [0, 1]$, $t \in \mathbb{R}$, $\zeta \in [0, 1]$, $\tau \in \mathbb{R}$):

$$\partial_t G_{\mathrm{ND}}^{(1)} - \Delta_z G_{\mathrm{ND}}^{(1)} = \delta(z - \zeta)\delta(t - \tau)\,,$$

$$\partial_z G_{\mathrm{ND}}^{(1)}|_{z=0} = 0\,, \quad G_{\mathrm{ND}}^{(1)}|_{z=1} = 0\,, \quad G_{\mathrm{ND}}^{(1)}|_{t<\tau} = 0\ :$$

$$G_{\mathrm{ND}}^{(1)}(z, t; \zeta, \tau) = H(t - \tau) \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{(2n+1)^2\pi^2}{4}(t-\tau)}\, \mathrm{C}_n(z)\, \mathrm{C}_n(\zeta)\,. \tag{B.2}$$

The functions $\mathrm{C}_n(z) := \sqrt{2} \cos \frac{(2n+1)\pi z}{2}$ ($n = 0, 1, 2, \ldots$) are the normalized eigenfunctions (and $\lambda_n = -\left[(2n + 1)\pi/2\right]^2$ are the corresponding eigenvalues) of the Sturm-Liouville problem $\frac{\mathrm{d}^2}{\mathrm{d}z^2} Z(z) = \lambda Z(z)$, $Z'(0) = 0$, $Z(1) = 0$, on the interval $z \in (0, 1)$ (see, e.g., the textbooks [17]).

(iii) Bessel functions (see, e.g., [20, Chapter 9]):

$$K_\nu(a) = \int_0^\infty \cosh(\nu w)\, \mathrm{e}^{-a \cosh w}\, \mathrm{d}w \qquad \text{for } |\arg a| < \frac{\pi}{2}\,, \tag{B.3}$$

$$I_0(a) = \frac{1}{\pi} \int_0^\pi \mathrm{e}^{\pm a \cos\theta}\, \mathrm{d}\theta\,, \tag{B.4}$$

$$\int_0^\infty K_0(aw)\, I_0(w)\, w\, \mathrm{d}w = \frac{1}{a^2 - 1} \qquad \text{for } a > 1\,. \tag{B.5}$$

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