Derivation of some Itô integrals

Proof that $\int_{t_0}^{t} B_s \, \mathrm{d}B_s = \frac{1}{2}\left(B_t^2 - B_{t_0}^2\right) - \frac{1}{2}(t - t_0)$

To compute the integral $\int_{t_0}^{t} B_s \, \mathrm{d}B_s$ in the Itô sense, we recall that it is defined as the mean square limit of

$$\sum_{i=0}^{n-1} B_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right) \qquad \text{as } \max_i |\Delta t_i| \to 0 \ ;$$

often this is denoted as (m.s.=“mean square”)

$$\int_{t_0}^{t} B_s \, \mathrm{d}B_s := \text{m.s.} \lim_{\max_i |\Delta t_i| \to 0} \sum_{i=0}^{n-1} B_{t_i} \left(B_{t_{i+1}} - B_{t_i}\right) \ .$$

Here we will explain this in more detail and will find $\int_{t_0}^{t} B_s \, \mathrm{d}B_s$.

Let $\mathbf{t}$ be a partition, $t_0 < t_1 < t_2 < \cdots < t_n = t$, of the interval $[t_0, t]$ into n parts (not necessarily of equal length): $[t_0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$, ..., $[t_{n-1}, t_n]$. For convenience, we set $B_i := B_{t_i}$, $\Delta B_i := B_{i+1} - B_i$, and $\|\mathbf{t}\| := \max_i |\Delta t_i|$. The partial sum corresponding to the partition $\mathbf{t}$ is

$$\begin{aligned}
S_{\mathbf{t}} &= \sum_{i=0}^{n-1} B_i \left(B_{i+1} - B_i\right) = \sum_{i=0}^{n-1} B_i \, \Delta B_i \\
&= \frac{1}{2} \sum_{i=0}^{n-1} \left[(B_i + \Delta B_i)^2 - B_i^2 - (\Delta B_i)^2\right] \\
&= \frac{1}{2} \sum_{i=0}^{n-1} (B_{i+1}^2 - B_i^2) - \frac{1}{2} \sum_{i=0}^{n-1} (\Delta B_i)^2 \\
&\quad \text{(telescoping sum!)} \\
&= \frac{1}{2}\left(B_t^2 - B_{t_0}^2\right) - \frac{1}{2} \sum_{i=0}^{n-1} (\Delta B_i)^2 \ . \qquad (1)
\end{aligned}$$

Now we will find the mean square limit of the last term. First note that, because of

$$\mathbb{E}\left[(\Delta B_i)^2\right] = \Delta t_i = t_{i+1} - t_i \ ,$$

we have

$$\mathbb{E}\left[\sum_{i=0}^{n-1} (\Delta B_i)^2\right] = \sum_{i=0}^{n-1} \mathbb{E}\left[(\Delta B_i)^2\right] = \sum_{i=0}^{n-1} \Delta t_i = t - t_0$$

(telescoping sum again!). So our guess is that

$$\sum_{i=0}^{n-1} (\Delta B_i)^2 \to t - t_0 \qquad \text{in mean square, as } \|\mathbf{t}\| \to 0 \ , \qquad (2)$$

i.e., that

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1} (\Delta B_i)^2 - (t - t_0)\right)^2\right] \to 0 \qquad \text{as } \|\mathbf{t}\| \to 0 \ .$$

Here is the proof:

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=0}^{n-1}(\Delta B_i)^2-(t-t_0)\right)^2\right]\\
&\qquad=\mathbb{E}\left[\left(\sum_{i=0}^{n-1}(\Delta B_i)^2\right)^2-2(t-t_0)\sum_{i=0}^{n-1}(\Delta B_i)^2+(t-t_0)^2\right]\\
&\qquad\left(\text{use the formula }\left(\sum_i a_i\right)^2=\sum_i a_i^2+2\sum_{i<j}a_ia_j\right)\\
&\qquad=\mathbb{E}\left[\sum_{i=0}^{n-1}(\Delta B_i)^4+2\sum_{i<j}(\Delta B_i)^2(\Delta B_j)^2-2(t-t_0)\sum_{i=0}^{n-1}(\Delta B_i)^2+(t-t_0)^2\right]\\
&\qquad=\sum_{i=0}^{n-1}\mathbb{E}\left[(\Delta B_i)^4\right]+2\sum_{i<j}\mathbb{E}\left[(\Delta B_i)^2(\Delta B_j)^2\right]-2(t-t_0)\sum_{i=0}^{n-1}\mathbb{E}\left[(\Delta B_i)^2\right]+(t-t_0)^2\ .
\end{aligned}
$$

Now recall that $\mathbb{E}\left[(\Delta B_i)^4\right]=3(\Delta t_i)^2$ and $\mathbb{E}\left[(\Delta B_i)^2\right]=\Delta t_i$; also, since the time intervals $[t_i,t_{i+1}]$ and $[t_j,t_{j+1}]$ do not overlap for $i\neq j$, the increments ΔB_i and ΔB_j are independent, hence

$$
\mathbb{E}\left[(\Delta B_i)^2(\Delta B_j)^2\right]=\mathbb{E}\left[(\Delta B_i)^2\right]\mathbb{E}\left[(\Delta B_j)^2\right]=(\Delta t_i)(\Delta t_j)\ .
$$

Plugging all these in the expression above, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=0}^{n-1}(\Delta B_i)^2-(t-t_0)\right)^2\right]\\
&\qquad=3\sum_{i=0}^{n-1}(\Delta t_i)^2+2\sum_{i<j}(\Delta t_i)(\Delta t_j)-2(t-t_0)\sum_{i=0}^{n-1}\Delta t_i+(t-t_0)^2\\
&\qquad=2\sum_{i=0}^{n-1}(\Delta t_i)^2+\left\{\sum_{i=0}^{n-1}(\Delta t_i)^2+2\sum_{i<j}(\Delta t_i)(\Delta t_j)\right\}-2(t-t_0)\sum_{i=0}^{n-1}\Delta t_i+(t-t_0)^2\\
&\qquad=2\sum_{i=0}^{n-1}(\Delta t_i)^2+\left\{\sum_{i=0}^{n-1}\Delta t_i\right\}^2-2(t-t_0)^2+(t-t_0)^2\\
&\qquad=2\sum_{i=0}^{n-1}(\Delta t_i)^2\leq 2(t-t_0)\,\|\mathbf{t}\|\to 0\qquad\text{as }\|\mathbf{t}\|\to 0\ ,
\end{aligned}
$$

which completes the proof of (2). Here we have used the inequality

$$
\sum_{i=0}^{n-1}(\Delta t_i)^2\leq (t-t_0)\,\|\mathbf{t}\|\ ,
$$

which is a form of Hölder's inequality (see problem 4.14.27(a) of the book).

So, we proved that

$$
\int_{t_0}^{t}B_s\,\mathrm{d}B_s=\frac{1}{2}\left(B_t^2-B_{t_0}^2\right)-\frac{1}{2}\,(t-t_0)\ \ ,
$$

which can also be written as

$$
\int B_t\,\mathrm{d}B_t=\frac{1}{2}B_t^2-\frac{1}{2}t\ ,\qquad\text{or as}\qquad \mathrm{d}\left(B_t^2\right)=2\,B_t\,\mathrm{d}B_t+\mathrm{d}t\ .
$$

Comments:

- Note that
$$\mathbb{E}\left[\int_{t_0}^{t} B_s \,\mathrm{d}B_s\right] = \frac{1}{2}\left[B_t^2 - B_{t_0}^2 - (t-t_0)\right] = \frac{1}{2}\left[t - t_0 - (t-t_0)\right] = 0 \ ,$$
which is also obvious from the definition of the stochastic integral because for the individual terms in the partial sum we have $\mathbb{E}\left[B_i\, \Delta B_i\right] = 0$ (since B_i and ΔB_i are independent and $\mathbb{E}[\Delta B_i] = 0$).

- The reason for the stochastic integral to be different from the ordinary Riemann-Stiltjes integral is that the increments $\Delta B_i = B_{i+1} - B_i$ have means of order $\sqrt{\Delta t_i}$, so that – in contrast to the ordinary integration! – terms of order of $(\Delta B_i)^2$ do not vanish on taking the limit $\|\mathbf{t}\| \to 0$.

Proof that $\int_{t_0}^{t} B_s^2 \,\mathrm{d}B_s = \frac{1}{3}\left(B_t^3 - B_{t_0}^3\right) - \int_{t_0}^{t} B_s \,\mathrm{d}s$

From the formula
$$B_{i+1}^3 = (B_i + \Delta B_i)^3 = B_i^3 + 3\, B_i^2\, \Delta B_i + 3\, B_i\, (\Delta B_i)^2 + (\Delta B_i)^3$$
we obtain for the partial sum
$$\begin{aligned}
\sum_{i=0}^{n-1} B_i^2\, \Delta B_i &= \frac{1}{3}\sum_{i=0}^{n-1}\left(B_{i+1}^3 - B_i^3\right) - \sum_{i=0}^{n-1} B_i\,(\Delta B_i)^2 - \frac{1}{3}\sum_{i=0}^{n-1}(\Delta B_i)^3 \\
&= \frac{1}{3}\left(B_t^3 - B_{t_0}^3\right) - \sum_{i=0}^{n-1} B_i\,(\Delta B_i)^2 - \frac{1}{3}\sum_{i=0}^{n-1}(\Delta B_i)^3 \ . \qquad (3)
\end{aligned}$$

Here we analyze the sums in the right-hand side of (3). Since
$$\mathbb{E}\left[\sum_{i=0}^{n-1} B_i\,(\Delta B_i)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}[B_i]\,\mathbb{E}\left[(\Delta B_i)^2\right] = \sum_{i=0}^{n-1}\mathbb{E}[B_i]\,\Delta t_i \ ,$$
we suspect that
$$\mathbb{E}\left[\sum_{i=0}^{n-1} B_i\,(\Delta B_i)^2\right] \to \int_{t_0}^{t} B_s\,\mathrm{d}s \qquad \text{in mean square} \qquad \text{as } n\to\infty \ . \qquad (4)$$
Let us prove that our guess (4) is correct – we use the formula $(\sum_i a_i)^2 = \sum_i a_i^2 + \sum_{i<j} a_i a_j$:
$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=0}^{n-1} B_i\left[(\Delta B_i)^2 - \Delta t_i\right]\right)^2\right] &= \sum_{i=0}^{n-1}\mathbb{E}\left[B_i^2\left((\Delta B_i)^2 - \Delta t_i\right)^2\right] \\
&\quad +2\sum_{i<j}\mathbb{E}\left[B_i\, B_j\left((\Delta B_i)^2 - \Delta t_i\right)\left((\Delta B_j)^2 - \Delta t_j\right)\right] \\
&= \sum_{i=0}^{n-1}\mathbb{E}\left[B_i^2\right]\mathbb{E}\left[\left((\Delta B_i)^2 - \Delta t_i\right)^2\right] \\
&\quad +2\sum_{i<j}\mathbb{E}\left[B_i\right]\,\mathbb{E}\left[B_j\right]\mathbb{E}\left[(\Delta B_i)^2 - \Delta t_i\right]\mathbb{E}\left[(\Delta B_j)^2 - \Delta t_j\right] \ ;
\end{aligned}$$

now we have $\mathbb{E}[B_i^2] = t_i$, $\mathbb{E}[(\Delta B_i)^2 - \Delta t_i] = \mathbb{E}[(\Delta B_i)^2] - \Delta t_i = 0$,

$$\mathbb{E}\left[\left((\Delta B_i)^2 - \Delta t_i\right)^2\right] = \mathbb{E}\left[(\Delta B_i)^4\right] - 2\mathbb{E}\left[(\Delta B_i)^2\right]\Delta t_i + (\Delta t_i)^2 = 3(\Delta t_i)^2 - 2(\Delta t_i)^2 + (\Delta t_i)^2 = 2\,(\Delta t_i)^2 \ ,$$

so that

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1} B_i\left[(\Delta B_i)^2 - \Delta t_i\right]\right)^2\right] \quad = \quad \sum_{i=0}^{n-1} t_i \cdot 2\ (\Delta t_i)^2 \leq 2\,t \sum_{i=0}^{n-1}(\Delta t_i)^2 \to 0 \qquad \text{as } n \to \infty \ .$$

Finally, the last sum in the right-hand side of (3) goes to 0 in mean square because $\mathbb{E}\left[(\Delta B_i)^3\right] = 0$, so each of the terms in this sum has zero expectation.

So, we proved that

$$\int_{t_0}^{t} B_s^2\,\mathrm{d}B_s = \frac{1}{3}\left(B_t^3 - B_{t_0}^3\right) - \int_{t_0}^{t} B_s\,\mathrm{d}s \ .$$

Equivalently, in the form of an indefinite integral,

$$\int B_t^2\,\mathrm{d}B_t = \frac{1}{3}B_t^3 - \int B_t\,\mathrm{d}t \ ,$$

or, in a differential notation,

$$\mathrm{d}\left(B_t^3\right) = 3\,B_t^2\,\mathrm{d}B_t + B_t\,\mathrm{d}t \ .$$

A general formula for $\int_{t_0}^{t} B_s^k\,\mathrm{d}B_s$

Here is the general formula for integrals of powers of the Wiener process:

$$\int_{t_0}^{t} W_s^k\,\mathrm{d}W_s = \frac{1}{k+1}\left(W_t^{k+1} - W_{t_0}^{k+1}\right) - \frac{k}{2}\int_{t_0}^{t} W_s^{k-1}\,\mathrm{d}s \ .$$

Here is the same formula in the form of an indefinite integral:

$$\int W_t^k\,\mathrm{d}W_t = \frac{1}{k+1}W_t^{k+1} - \frac{k}{2}W_t^{k-1}\,\mathrm{d}t \ ,$$

and in a differential notation:

$$\mathrm{d}\left(W_t^k\right) = k\,W_t^{k-1}\,\mathrm{d}W_t + \frac{k(k-1)}{2}\,W_t^{k-2}\,\mathrm{d}t \ .$$