MATH 5763 Homework 9 Due Thu, 4/25/2019

Problem 1. [Integrated flip-flop process]

This problem is a continuation of Problem 4 from Homework 8. There you studied the flip-flop process $\{X_t\}_{t\geq 0}$ and computed several quantities related to this process; in particular, in part (b) of that problem, you found that $\mathbb{E}[X_t] = \frac{1}{2}$ and $\mathbb{E}\left[X_u X_v\right] = \frac{1}{4}\left(1 + \mathrm{e}^{-2\lambda(v-u)}\right)$ for $v > u$.

(a) Define the random process

$$Y_t := \int_0^t X_u \, \mathrm{d}u \ , \qquad t \geq 0 \ .$$

Show that $\mathbb{E}[Y_t] = \frac{t}{2}$ and $\mathbb{E}[Y_t^2] = \frac{t^2}{4} + \frac{t}{4\lambda} - \frac{1}{8\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right)$.

Hint: The expectation of the integral over t is equal to the integral over t of the expectation of the integrand, so that

$$\mathbb{E}[Y_t] = \mathbb{E}\left[\int_0^t X_u \, \mathrm{d}u\right] = \int_0^t \mathbb{E}\left[X_u\right] \mathrm{d}u = \cdots \ .$$

For $\mathbb{E}[Y_t^2]$, you can write

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \mathbb{E}\left[\left(\int_0^t X_u \, \mathrm{d}u\right)\left(\int_0^t X_v \, \mathrm{d}v\right)\right] = \mathbb{E}\left[\int_0^t \int_0^t X_u X_v \, \mathrm{d}u \, \mathrm{d}v\right] \\
&= \int_0^t \left(\int_0^u \mathbb{E}\left[X_u X_v\right] \mathrm{d}v\right) \mathrm{d}u + \int_0^t \left(\int_0^v \mathbb{E}\left[X_u X_v\right] \mathrm{d}u\right) \mathrm{d}v \\
&\qquad \text{[where we split the square } [0,t] \times [0,t] \text{ in two parts]} \\
&= 2 \int_0^t \left(\int_0^v \mathbb{E}\left[X_u X_v\right] \mathrm{d}u\right) \mathrm{d}v \ ;
\end{aligned}$$

now notice that in the last integral the integration is over $0 \leq u \leq v \leq t$, and recall the expression for $\mathbb{E}\left[X_u X_v\right]$.

(b) Find $\operatorname{var} Y_t$ for the random process defined in part (e). Does $\operatorname{var} Y_t$ behave reasonably in the limit $t \to 0^+$? Explain.

Problem 2. [Brownian motion as a limit of a symmetric simple random walk]

Let $X = \{X_t : t \geq 0\}$ be a time-homogeneous continuous-time Markov process with state space $\mathcal{X} = \eta\mathbb{Z}$, where $\eta\mathbb{Z}$ is a shorthand notation for the set of all integer multiples of η:

$$\eta\mathbb{Z} := \{\ldots, -2\eta, -\eta, 0, \eta, 2\eta, \ldots\} \ .$$

The process X is allowed to jump "up" or "down" by η with equal probabilities (like in the case of a symmetric simple random walk). Let the intensity of the process X be τ, i.e.,

$$p_{jk}(h) := \mathbb{P}\big(X(t+h) = k\eta \,\big|\, X_t = j\eta\big) = \begin{cases} \tau h + o(h) & \text{for } k = j \pm 1 \ , \\ 1 - 2\tau h + o(h) & \text{for } k = j \ , \\ o(h) & \text{otherwise} \ . \end{cases}$$

(a) Show that the probabilities $p_k(t) := \mathbb{P}(X_t = k\eta)$ satisfy the system of ODEs

$$p_k'(t) = \tau \left[p_{k-1}(t) - 2p_k(t) + p_{k+1}(t)\right] .$$

(b) Use the system of ODEs for the probabilities $p_k(t)$ to show that the characteristic function

$$\phi(\xi, t) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi X_t}\right] = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\xi k\eta}\, p_k(t)$$

satisfies the equation $\dfrac{\partial \phi}{\partial t} = \tau\left(\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}\right)\phi.$

(c) Assume that at $t = 0$, the process was at 0 (i.e., $X_0 = 0$). What does this imply for $\phi(\xi, 0)$? Solve the equation for $\phi(\xi, t)$ derived in part (b) with the initial condition you just found.

Hint: Although the equation for $\phi(\xi, t)$ derived in (b) is about a function of ξ and t, it does not contain ξ-derivatives, so you can solve it simply as an ordinary differential equation treating ξ as a fixed number. You will obtain $\phi(\xi, t) = \exp\left\{\tau\left(\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}\right)\right\}$, but I want to see your calculations.

(d) Now let the "spatial step-size" η of the process go to zero, and the "temporal intensity" τ of the process go to infinity, in such a way that $2\eta^2\tau \to 1$. Compare the expression for $\phi(\xi, t)$ (obtained in part (c)) in this limit with the characteristic function $\phi_{\mathrm{N}(\mu,\sigma^2)}(\xi) = \mathrm{e}^{\mathrm{i}\mu\xi - \frac{1}{2}\sigma^2\xi^2}$ of a normal random variable with mean μ and variance σ^2. What can you conclude about the distribution of the random variable X_t in the limit $\eta \to 0$, $\tau \to \infty$, $2\eta^2\tau \to 1$?

Hint: To perform the limiting transition, you can expand the expression $\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}$ (which will be part of your result for $\phi(\xi, t)$) in a Taylor series with respect to η around the point $\eta = 0$, and, after the obvious cancellations, you will obtain

$$\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta} = -\eta^2\xi^2 + o(\xi^2) .$$

Problem 3. [First passage time of a Wiener process]

Let $\{B(t)\}_{t\geq 0}$ be a standard Wiener process (for which $B(t) \sim \mathrm{N}(0, t)$). Let $m > 0$, fix $t > 0$, and consider the event $\{B(t) > m\}$. Since $B(s)$ is a continuous function for $0 \leq s \leq t$, and $B(0) = 0$, the Intermediate Value Theorem implies that if the event $B(t) > m$ occurs, then we have $B(s) = m$ for at least one $s \in [0, t]$. It is natural to be interested in the moment when the Wiener process reaches position m for the first time, so define

$$T_m := \inf\{s > 0 : B(s) = m\} .$$

Set

$$R(s) := \begin{cases} B(s) & \text{for } s < T_m \ , \\ 2m - B(s) & \text{for } s \geq T_m \ , \end{cases}$$

which may be envisaged as reflecting the portion of the path after T_m with respect to the horizontal line $\{y = m\}$. This construction – reminiscent of the Reflection Principle from Food for Thought Problem 2 of Homework 5 (also used in Problem 4 of Homework 6) – is represented in the figure below; note that by the definition of T_m, $R(t) \leq m$ when $B(t) \geq m$.

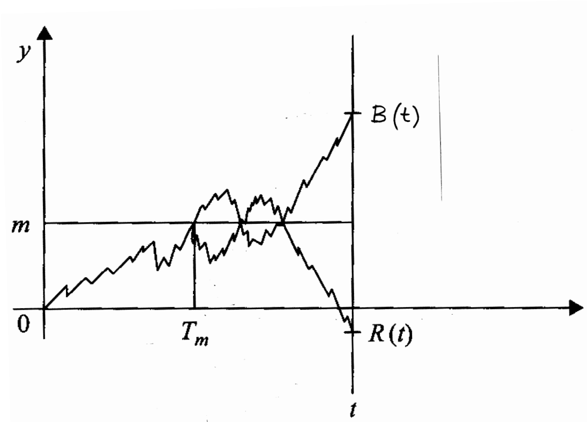

Figure 1: Sketch illustrating the Reflection Principle.

The argument of the Reflection Principle first asserts that the original and the reflected paths are equally likely, because of the symmetry of the normal distribution around its mean (here we are using the fact that $B(t) \sim N(0,t)$). Second, it observes that reflection is a one-to-one transformation, and finally it claims that we therefore have

$$\mathbb{P}(T_m \le t,\, B(t) > m) = \mathbb{P}(T_m \le t,\, B(t) < m)\ . \tag{1}$$

This observation yields a remarkable way to find the distribution of T_m, done in the steps below.

(a) How are the events $\{B(t) > m\}$ and $\{T_m \le t\}$ related? Explain why this implies that

$$\mathbb{P}(T_m \le t,\, B(t) > m) = \mathbb{P}(B(t) > m)\ . \tag{2}$$

(b) Note that the events $\{B(t) > m\}$ and $\{B(t) < m\}$ form a partition of the sample space. Use this fact together with the Reflection Principle (1) and the equality (2) to show that

$$\mathbb{P}(T_m \le t) = \mathbb{P}(|B(t)| > m) = \sqrt{\frac{2}{\pi t}} \int_m^\infty \mathrm{e}^{-x^2/(2t)}\, \mathrm{d}x = \int_0^t \frac{|m|}{\sqrt{2\pi y^3}}\, \mathrm{e}^{-m^2/(2y)}\, \mathrm{d}y\ .$$

Here we wrote absolute value of m in the final expression to make the formula applicable to any $m \in \mathbb{R}$, not just to positive values of m.

Hint: Here is a plan:

- use that $\{T_m \le t\} = \{T_m \le t, B(t) > m\} \cup \{T_m \le t, B(t) < m\}$ (why?),
- apply (1) and (2),
- use that, by symmetry, $2\mathbb{P}(B(t) > m) = \mathbb{P}(|B(t)| > m)$ for any $m > 0$,
- express $\mathbb{P}(|B(t)| > m)$ as an integral by using that $B(t) \sim \mathrm{N}(0,t)$,
- change variables in the integral you wrote to obtain the desired result.

(c) What is the p.d.f. of the random variable T_m?

Food for Thought. One can show that $\mathbb{P}(T_m < \infty) = 1$ by showing that $\int_0^\infty f_{T_m}(x)\, \mathrm{d}x = 1$. Think about the meaning of this fact. There is no need to do this calculation or to write anything about this in your write-up.

Food for Thought Problem 1. [Martingales]
Let $\{B_t\}_{t\geq 0}$ be a standard Wiener process, and $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration of σ-algebras generated by $\{B_t\}_{t\geq 0}$. Prove that the process $Z_t := tW_t - \displaystyle\int_0^t W_r \,\mathrm{d}r$ is a martingale with respect to this filtration. Do not worry about the integrability condition – all you have to show is that $\mathbb{E}\left[tW_t - \displaystyle\int_0^t W_r \,\mathrm{d}r \;\middle|\; \mathcal{F}_s \right] = sW_s - \displaystyle\int_0^s W_r \,\mathrm{d}r.$

Food for Thought Problem 2. [For lovers of σ-algebras]
Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{A}, \mathbb{L})$, where $\mathcal{A}$ is the Borel σ-algebra on $[0,1]$, and $\mathbb{L}$ is the Lebesgue measure on $[0,1]$; simply speaking, this means that you know the probability (i.e., measure) of each interval $(a,b) \subseteq [0,1]$, and it is equal to its length, $\mathbb{L}((a,b)) = b-a$ (the probabilities of $(a,b]$, $[a,b)$, and $[a,b]$ are also $b-a$). If $X : [0,1] \to \mathbb{R}$ is a random variable, then $\int_A X(\omega)\,\mathrm{d}\mathbb{L}(\omega) = \int_A X(\omega)\,\mathrm{d}\omega$ (for any $A \in \mathcal{A}$) is the ordinary integral.
Let the random variables X and Y, both on $([0,1], \mathcal{A}, \mathbb{L})$ be defined as follows:

$$X(\omega) = \omega^2 \quad \forall \omega \in [0,1] \; ; \qquad Y(\omega) = \begin{cases} \frac{1}{5} & \text{for } \omega \in [0, \frac{1}{3}] \; , \\ \frac{1}{2} & \text{for } \omega \in (\frac{1}{3}, 1] \; . \end{cases}$$

(a) Find explicitly the σ-algebra $\sigma(Y)$ generated by the random variable Y.

(b) Find $\mathbb{E}[X]$ directly from the definition of expectation, $\mathbb{E}[X] = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{L}(\omega)$.

Remark: Usually the probability measure is not so easy to deal with, so one computes $\mathbb{E}[X]$ by changing variables from ω to $x = X(\omega)$, and the formula becomes $\mathbb{E}[X] = \int_{\mathbb{R}} x\,\mathrm{d}F_X(x)$, where $F_X : \mathbb{R} \to [0,1]$ is the c.d.f. of X, defined as $F_X(x) = \mathbb{P}(\{X \leq x\}) = \mathbb{P}\left(X^{-1}((-\infty, x])\right)$. But in this case $\mathrm{d}\mathbb{L}(\omega) = \mathrm{d}\omega$, so that direct computation of $\mathbb{E}[X]$ is straightforward.

(c) Find the conditional expectation $\mathbb{E}[X|Y]$.

Hint: $\mathbb{E}[X|Y] = \mathbb{E}\big[X\big|[0,\frac{1}{3}]\big]\,\chi_{[0,\frac{1}{3}]} + \mathbb{E}\big[X\big|(\frac{1}{3},1]\big]\,\chi_{(\frac{1}{3},1]} = \frac{1}{27}\,\chi_{[0,\frac{1}{3}]} + \frac{13}{27}\,\chi_{(\frac{1}{3},1]}$.

(d) Let Z be a random variable on $([0,1], \mathcal{A}, \mathbb{L})$ defined as

$$Z(\omega) = \begin{cases} \frac{1}{5} & \text{for } \omega \in [0, \frac{1}{3}] \; , \\ \omega & \text{for } \omega \in (\frac{1}{3}, 1] \; . \end{cases}$$

What is the σ-algebra $\sigma(Z)$ generated by the random variable Z? (Since $\sigma(X)$ contains infinitely many sets, just describe them in words.)

(e) Find the conditional expectation $\mathbb{E}[X|Z]$.

(f) Compute $\mathbb{E}[\mathbb{E}[X|Z]]$ and $\mathbb{E}[\mathbb{E}[X|Z]]$; compare with $\mathbb{E}[X]$ from part (b).