MATH 5763 Homework 9 Due Tue, 11/24/2015

Problem 1. Let $\{B(t)\}_{t\geq 0}$ be a standard Wiener process (for which $B(t) \sim N(0,t)$). Let $m > 0$, fix $t > 0$, and consider the event $\{B(t) > m\}$. Since $B(s)$ is a continuous function for $0 \leq s \leq t$, and $B(0) = 0$, the Intermediate Value Theorem implies that if the event $B(t) > m$ occurs, then we have $B(s) = m$ for at least one $s \in [0,t]$. It is natural to be interested in the moment when the Wiener process reaches position m for the first time, so define

$$T_m := \inf\{s > 0 : B(s) = m\} \ .$$

Set

$$R(s) := \begin{cases} B(s) & \text{for } s < T_m \ , \\ 2m - B(s) & \text{for } s \geq T_m \ , \end{cases}$$

which may be envisaged as reflecting the portion of the path after T_m with respect to the horizontal line $\{y = m\}$. This construction – reminiscent of the Reflection Principle from Food for Thought Problem 2 of Homework 5 – is represented in the figure below; note that by the definition of T_m, $R(t) \leq m$ when $B(t) \geq m$.

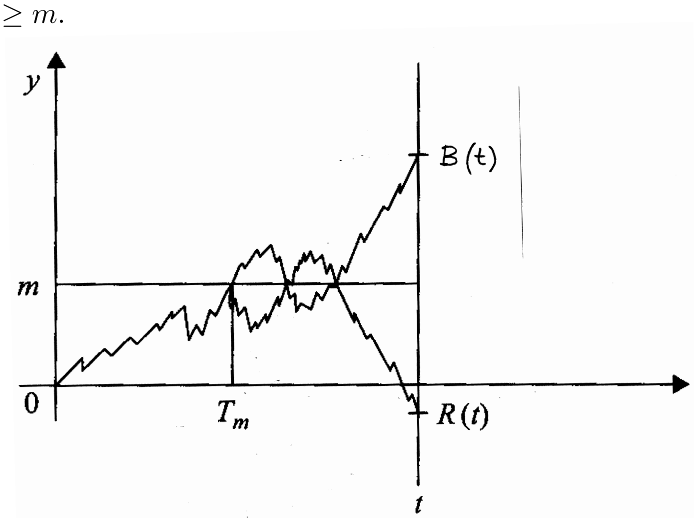

Figure 1: Sketch illustrating the Reflection Principle.

The argument of the Reflection Principle first asserts that the original and the reflected paths are equally likely, because of the symmetry of the normal distribution around its mean (here we are using the fact that $B(t) \sim N(0,t)$). Second, it observes that reflection is a one-to-one transformation, and finally it claims that we therefore have

$$\mathbb{P}(T_m \leq t,\, B(t) > m) = \mathbb{P}(T_m \leq t,\, B(t) < m) \ . \tag{1}$$

This observation yields a remarkably neat way to find the distribution of T_m, done in the steps below.

(a) How are the events $\{B(t) > m\}$ and $\{T_m \leq t\}$ related? Explain why this implies that

$$\mathbb{P}(T_m \leq t,\, B(t) > m) = \mathbb{P}(B(t) > m) \ . \tag{2}$$

(b) Note that the events $\{B(t) > m\}$ and $\{B(t) < m\}$ form a partition of the sample space. Use this fact together with the Reflection Principle (1) and the equality (2) to show that

$$\mathbb{P}(T_m \le t) = \mathbb{P}(|B(t)| > m) = \sqrt{\frac{2}{\pi t}} \int_m^\infty \mathrm{e}^{-x^2/(2t)}\, \mathrm{d}x = \int_0^t \frac{|m|}{\sqrt{2\pi y^3}}\, \mathrm{e}^{-m^2/(2y)}\, \mathrm{d}y \; .$$

Here we wrote absolute value of m in the final expression to make the formula applicable to any $m \in \mathbb{R}$, not just to positive values of m.

(c) What is the p.d.f. of the random variable T_m?

Food for Thought. One can show that $\mathbb{P}(T_m < \infty) = 1$ by showing that $\int_0^\infty f_{T_m}(x)\, \mathrm{d}x = 1$. Think about the meaning of this fact. There is no need to do this calculation or to write anything about this in your write-up.

Problem 2. Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion (i.e., $B(t) \sim N(0,t)$). Define the *integrated Brownian motion*

$$Z(t) := \int_0^t B(u)\, \mathrm{d}u \; .$$

One can prove that the integrated Brownian motion is a Gaussian process (you do not need to prove this). The increments of the process Z, however, are not independent, as you will show below.

(a) For $0 \le s < t$, use that $\mathbb{E}[B(u)\, B(v)] = \min\{u, v\}$, to compute the autocorrelation function $R_Z(t,s) = \mathbb{E}[Z(t)\, Z(s)]$. The calculation goes like this:

$$\begin{aligned} R_Z(t,s) &= \mathbb{E}[Z(t)\, Z(s)] \\ &= \mathbb{E}\left[\int_0^t B(u)\, \mathrm{d}u \int_0^s B(v)\, \mathrm{d}v\right] \\ &= \int_0^t \int_0^s \mathbb{E}\left[B(u)\, B(v)\right]\, \mathrm{d}v\, \mathrm{d}u \\ &= \int_0^t \left(\int_0^s \min\{u,v\}\, \mathrm{d}v\right) \mathrm{d}u \; , \end{aligned}$$

and perform the integration over the rectangle $(u,v) \in [0,t] \times [0,s]$ in the (u,v)-plane (remember that $t > s$); use that $\min\{u,v\} = u$ above the diagonal $v = u$, and $\min\{u,v\} = v$ below the diagonal. Be careful with the limits of integration. The result is $R_Z(t,s) = s^2\left(\frac{t}{2} - \frac{s}{6}\right)$.

(b) Use the result from (a) to compute $\mathrm{Cov}\,(Z(t) - Z(s), Z(s))$.

(c) Use your result from part (b) to decide whether the increments $(Z(t) - Z(s))$ and $(Z(s) - Z(0))$ are independent.

Problem 3. Use the fact that the Wiener process has independent increments to show that $W(t)$ and $(W(t)^2 - t)$ are martingales. You have to prove only that $\mathbb{E}[W(t)|W(s)] = W(s)$ and that $\mathbb{E}[W(t)^2 - t|W(s)] = W(s)^2 - s$ for any $0 \le s < t$.