MATH 5463 Homework 9 Due Thu, Apr 23

Problem 8.2/8 from Folland's book.
Hint: This problem is not difficult if you understand what you are asked to do. Perhaps the proof of Proposition 8.8 can help you.

Additional problem 1. Let f and g be real-valued functions in $L^2([0,1],m)$, and let the function f have zero average, i.e., satisfy $\int_0^1 f(x)\,\mathrm{d}x = 0$. Show that

$$\left(\int_0^1 f(x)\,g(x)\,\mathrm{d}x\right)^2 \le \left(\int_0^1 |f(x)|^2\,\mathrm{d}x\right)^2 \left[\int_0^1 |g(x)|^2\,\mathrm{d}x - \left(\int_0^1 g(x)\,\mathrm{d}x\right)^2\right].$$

Hint: Let $a = \int_0^1 g(x)\,\mathrm{d}x$, and note that $\int_0^1 f(x)\,g(x)\,\mathrm{d}x = \int_0^1 f(x)\,[g(x)-a]\,\mathrm{d}x$ (why?).

Additional problem 2. Let $p \in [1,\infty)$. All parts of this problem refer to the Lebesgue measure on $\mathbb{R}$.

(a) Give an example where the sequence $\{f_n\}$ converges to f pointwise, $\|f_n\|_p \le M$ for any $n \in \mathbb{N}$ and some constant M, and $\|f_n - f\|_p \nrightarrow 0$.

(b) If the sequence $\{f_n\}$ converge to f pointwise and $\|f_n\|_p \to A < \infty$, what can you conclude about $\|f\|_p$? Justify your conclusions.
Hint: Use Fatou's Lemma.

(c) Show that if the sequence $\{f_n\}$ converges to f pointwise and $\|f_n\|_p \to \|f\|_p$, then $\|f_n - f\|_p \to 0$.
Hint: Define the function sequence $\{g_n\}$ by $g_n = 2^p\left(|f_n|^p + |f|^p\right) - |f_n - f|^p$, give a (detailed) proof that $g_n \ge 0$, find the pointwise limit of g_n, and then use Fatou's Lemma.

Additional problem 3. Let $f : \mathbb{R} \to [0,\infty)$ be a measurable function, and $0 < r < \infty$. Prove that, for any measurable set E with non-zero measure,

$$\left(\frac{1}{\mu(E)}\int_E f\,\mathrm{d}\mu\right)^{-1} \le \left(\frac{1}{\mu(E)}\int_E \frac{1}{f^r}\,\mathrm{d}\mu\right)^{1/r}.$$

Hint: Set $p = 1 + \frac{1}{r} > 1$, note that $\mu(E) = \int_E 1\,\mathrm{d}\mu = \int_E f^{\frac{1}{p}}\,f^{-\frac{1}{p}}\,\mathrm{d}\mu$, and apply Hölder inequality.