

Problem 8.2/8 from Folland's book.

Hint: This problem is not difficult if you understand what you are asked to do. Perhaps the proof of Proposition 8.8 can help you.

Additional problem 1. Let f and g be real-valued functions in $L^2([0, 1], m)$, and let the function f have zero average, i.e., satisfy $\int_0^1 f(x) dx = 0$. Show that

$$\left(\int_0^1 f(x) g(x) dx \right)^2 \leq \left(\int_0^1 |f(x)|^2 dx \right)^2 \left[\int_0^1 |g(x)|^2 dx - \left(\int_0^1 g(x) dx \right)^2 \right].$$

Hint: Let $a = \int_0^1 g(x) dx$, and note that $\int_0^1 f(x) g(x) dx = \int_0^1 f(x) [g(x) - a] dx$ (why?).

Additional problem 2. Let $p \in [1, \infty)$. All parts of this problem refer to the Lebesgue measure on \mathbb{R} .

- (a) Give an example where the sequence $\{f_n\}$ converges to f pointwise, $\|f_n\|_p \leq M$ for any $n \in \mathbb{N}$ and some constant M , and $\|f_n - f\|_p \not\rightarrow 0$.
- (b) If the sequence $\{f_n\}$ converge to f pointwise and $\|f_n\|_p \rightarrow A < \infty$, what can you conclude about $\|f\|_p$? Justify your conclusions.

Hint: Use Fatou's Lemma.

- (c) Show that if the sequence $\{f_n\}$ converges to f pointwise and $\|f_n\|_p \rightarrow \|f\|_p$, then $\|f_n - f\|_p \rightarrow 0$.

Hint: Define the function sequence $\{g_n\}$ by $g_n = 2^p (|f_n|^p + |f|^p) - |f_n - f|^p$, give a (detailed) proof that $g_n \geq 0$, find the pointwise limit of g_n , and then use Fatou's Lemma.

Additional problem 3. Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a measurable function, and $0 < r < \infty$. Prove that, for any measurable set E with non-zero measure,

$$\left(\frac{1}{\mu(E)} \int_E f d\mu \right)^{-1} \leq \left(\frac{1}{\mu(E)} \int_E \frac{1}{f^r} d\mu \right)^{1/r}.$$

Hint: Set $p = 1 + \frac{1}{r} > 1$, note that $\mu(E) = \int_E 1 d\mu = \int_E f^{\frac{1}{p}} f^{-\frac{1}{p}} d\mu$, and apply Hölder inequality.