MATH 4163 **Homework 9** **Due Thu, 04/05/2018**

Problem 1. [Sound waves in a pipe]

The pipe that I brought to class had length 65 cm. Assume that one end of the pipe is closed while the other is open.

(a) Draw a (neat!) picture of the pipe and the first several waves whose lengths are allowed by the boundary conditions. Your picture should make clear what the values of the three largest allowed wavelengths are; write down these values. No calculations are needed in this part of the problem.

(b) Write down the functions $X_n(x)$ satisfying the appropriate boundary conditions and find the expressions for the allowed wavelengths λ_n. Of course, you will obtain the same results as the ones that you found pictorially in part (a).

(c) The speed of sound in normal conditions is usually taken to be 345 m/s. Find the numerical values of the four lowest allowed frequencies.

(d) From the list of frequencies corresponding to the different musical notes which can be found, e.g., at

`https://pages.mtu.edu/~suits/notefreqs.html`

find the notes that are closest to the first four frequencies in the pipe.

Problem 2. [Wave equation with homogeneous Neumann BCs]

In this problem you will solve the following BVP for the wave equation with homogeneous (i.e., zero) Neumann BCs:

$$\begin{aligned} &u_{tt}(x,t) = c^2 u_{xx}(x,t) \ , \qquad x \in [0,L] \ , \quad t \in [0,\infty) \ , \\ &u_x(0,t) = 0 \ , \quad u_x(L,t) = 0 \ , \\ &u(x,0) = f(x) \ , \quad u_t(x,0) = g(x) \ , \end{aligned} \tag{1}$$

where you can assume that you know the Fourier cosine or Fourier sine series (whichever one you need) of the functions f and g giving the initial position and the initial velocity of the points of the string.

The meaning of the BCs is the following: the ends of the string are attached to small rings (with negligible mass) that can slide without friction on two thin vertical rods, as shown in Figure 1.

(a) Set $u(x,t) = X(x)\,T(t)$ and obtain the equations for T and X coming from the separation of variables.

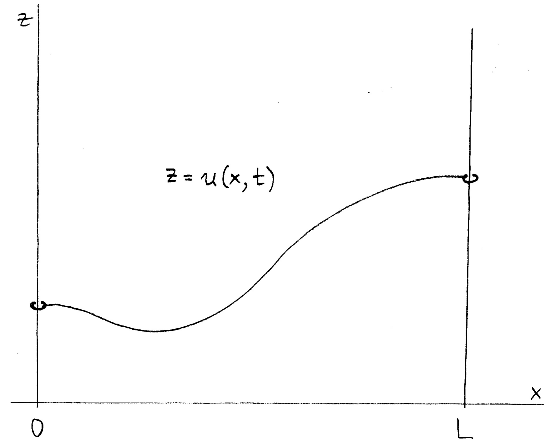

Figure 1: String with homogeneous Neumann BCs.

(b) Solve the BVP for X coming from the separation of variables and the BCs.

Hint: You have done this in previous homework problems. You should obtain a set of functions X_n where $n = 0, 1, 2, \ldots$.

(c) Write down the equations for the functions T_n and find their general solutions. You have to consider the cases $n = 0$ and $n \in \mathbb{N}$ separately. For each n, the function T_n must have two arbitrary constants, say A_n and B_n (because the equations for T_n are of second order).

(d) Write the solution $u(x,t)$ of the BVP (1) as a superposition of the functions $u_n(x,t) = X_n(x)\,T_n(t)$, and apply the ICs to determine the constants A_n and B_n in terms of the sine or cosine expansions of the functions f and g from the initial conditions.

(e) The general theory says that the each solution of the wave equation in one spatial dimension can be written in the form $u(x,t) = F(x+ct) + G(x-ct)$, where F and G are functions of one variable. The terms $F(x+ct)$ and $G(x-ct)$ describe waves moving to the left, resp. to the right, with speed c. Write the solution of the BVP (1) from part (d) in this form and find the concrete expressions for F and G (as infinite series).

(f) The expression $z_{\mathrm{CM}}(t) = \dfrac{1}{L}\displaystyle\int_0^L u(x,t)\,\mathrm{d}x$ gives the z-coordinate of the center of mass of the string at time t. Find $z_{\mathrm{CM}}(t)$ for the solution you obtained in part (d). The computation is very simple because $\displaystyle\int_0^L \cos\frac{n\pi x}{L}\,\mathrm{d}x = 0$, so that most terms in $u(x,t)$ will not contribute to $z_{\mathrm{CM}}(t)$. Look at the ICs and give a physical interpretation of your result for $z_{\mathrm{CM}}(t)$. Use this to give a physical interpretation of the Fourier sine/cosine coefficient(s) of f and g that occur in the expression for $z_{\mathrm{CM}}(t)$.

Problem 3. [D'Alembert formula; energy density of the wave]
Consider the following initial value problem for the wave equation on $\mathbb{R}$:

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t) \ , \quad x \in \mathbb{R} \ , \\ u(x,0) &= \arctan x \ , \quad u_t(x,0) = \frac{c}{1+x^2} \ . \end{aligned} \tag{2}$$

(a) Solve the initial value problem (2) by using D'Alembert's formula.
Hint: Your result will be very simple, and will correspond to a wave moving in only one direction.

(b) Recall that the linear density of the kinetic energy of the string is $\frac{\rho u_t^2}{2}$ while the linear density of the potential energy is $\frac{\tau u_x^2}{2}$. Compute the linear densities of the kinetic and the potential energy of the string; use that the speed of the wave is given by $c = \sqrt{\frac{\tau}{\rho}}$ to express your results without using c.

(c) Compute the total energy of the string, i.e., the integral of the total linear energy density. Discuss briefly the physical meaning of your result.

Problem 4. [Waves in a vertically hanging string]
In this problem you will study the waves in a string of length L that is hanging vertically down from a fixed point on the ceiling. Choose the origin of the coordinate system to be at the point where the lower end of the string is when the string is hanging at rest, i.e., the origin is at a distance L under the point where the string is suspended. Let the positive direction of the x-axis be vertically upward. We assume that the string moves in the (x,y)-plane, and its position at time t is described by the equation $y = u(x,t)$, as shown in Figure 2.

As we know, that the motion of the string is governed by the PDE

$$u_{tt}(x,t) = g\,\frac{\partial}{\partial x}\big[x\,u_x(x,t)\big] \ , \qquad x \in [0,L] \ , \quad t > 0 \ , \tag{3}$$

where $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the free-fall acceleration. Physical reasoning shows that the free end of the string (at $x = 0$) satisfies a homogeneous Neumann BC,

$$u_x(0,t) = 0 \ . \tag{4}$$

(a) What BC does the string satisfy at $x = L$ (i.e., at the suspension point)?

(b) We will change the variable x to a new variable, s, by

$$s = \widetilde{s}(x) := 2\sqrt{\frac{x}{g}} \ , \quad \text{or, equivalently, by} \quad x = \widetilde{x}(s) = \frac{g}{4}\,s^2 \ . \tag{5}$$

Here $\widetilde{s}$ and $\widetilde{x}$ are functions given by the explicit expressions above.

We define a new function, $v(s,t)$, by

$$v(s,t) := u(x,t)|_{x=\widetilde{x}(s)} = u(\widetilde{x}(s),t) \ , \tag{6}$$

or, equivalently, by

$$u(x,t) = v(s,t)|_{s=\widetilde{s}(x)} = v(\widetilde{s}(x),t) \ .$$

To write the PDE (3) in terms of the function $v(s,t)$, we compute:

$$u_t(x,t) = \frac{\partial}{\partial t} v(\widetilde{s}(x),t) = v_t(\widetilde{s}(x),t) \ ,$$

$$u_x(x,t) = \frac{\partial}{\partial x} v(\widetilde{s}(x),t) = v_s(\widetilde{s}(x),t) \frac{\mathrm{d}\widetilde{s}}{\mathrm{d}x} = \frac{1}{\sqrt{gx}} v_s(\widetilde{s}(x),t) \ .$$

Compute $u_{tt}(x,t)$ and $\dfrac{\partial}{\partial x}\big[x\, u_x(x,t)\big]$ in terms of the function $v(x,t)$ and its partial derivatives (please write your calculations in detail). Plug these expressions in the PDE (3) to show that the function $v(s,t)$ satisfy the PDE

$$v_{tt}(s,t) = \frac{1}{s} v_s(x,t) + v_{ss}(s,t) \ . \tag{7}$$

Write down the interval where the new variable, s, is taking values, as well as the BCs at the two ends of the string (coming from (4) and what you found in part (a)).

(c) Separate variables in the PDE (7) as usual: set $v(s,t) = S(s)\,T(t)$. The sign of the separation of variables constant must be such that the function $T(t)$ must be oscillatory, i.e., $T(t)$ must satisfy the ODE $T''(t) + \lambda^2 T(t) = 0$ whose general solution is $T(t) = C_1 \cos(\lambda t) + C_2 \sin(\lambda t)$. Here λ is a constant which can be assumed positive without loss of generality: $\lambda > 0$. As usual, the constant λ can take only a discrete set of values that will be found later. What ODE does the function $S(s)$ satisfy?

(d) Recall from Problem 5 of Homework 3 that the ODE

$$w''(z) + \tfrac{1}{z}\, w'(z) + \left(1 - \tfrac{n^2}{z^2}\right) w(z) = 0, \quad n = 0, 1, 2, 3, \dots \ , \tag{8}$$

is called the *Bessel differential equation.* The general solution of (8) is usually written as

$$w(z) = A\, J_n(z) + B\, Y_n(z) \ ,$$

where the functions $J_n(z)$ and $Y_n(z)$ called respectively *Bessel functions* and *Neumann functions* (or Bessel functions of first, resp., second kind). The graphs of the first several Bessel and Neumann functions are shown in Figure 3. Here are several important facts to notice (some of which will be needed below): $J_0(0) = 1$, $J_0'(0) = 0$, $J_n(0) = 0$ for $n = 1, 2, 3, \dots$, while all functions $Y_n(x)$ tend to $-\infty$ as $x \to 0^+$.

Change the variable s to $z := \lambda s$, and the unknown function $S(s)$ to $Z(z) := S\left(\frac{z}{\lambda}\right)$ to show that the function $Z(z)$ satisfy the Bessel ODE with $n = 0$, so that $Z(z)$ is a linear combination of the functions $J_0(z)$ and $Y_0(z)$. Then change the independent variable back to s to find $S(s)$.

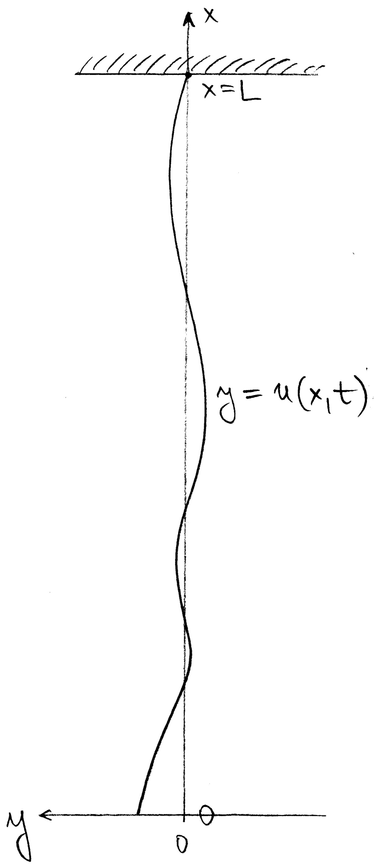

Figure 2: Hanging string.

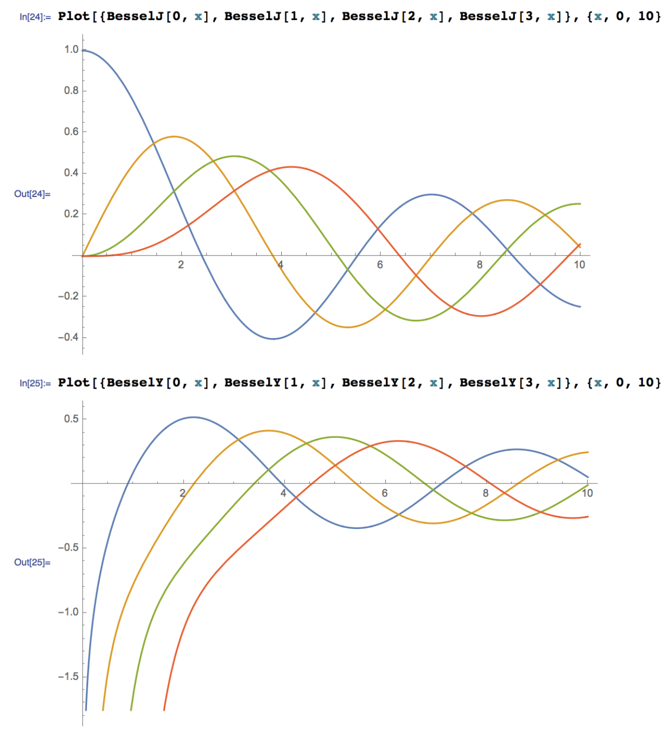

Figure 3: Plots of the Bessel functions J_n and the Neumann functions Y_n for $n = 0, 1, 2, 3$.

(e) Some of the functions obtained in part (d) behave non-physically. Which ones? (Look at the graphs in Figure 3.) Write down the expression for $S(s)$ if we want it to behave in a physically reasonable way.

(f) Does the function found in part (e) satisfy the BC at $x = 0$? Discuss briefly.

(g) Impose the BC at $x = L$, and obtain the values that the constant λ can take. Let ξ_{0k} ($k = 1, 2, 3, \ldots$) be the kth zero of $J_0(\xi)$, i.e., $J_0(\xi_{0k}) = 0$, ordered so that $\xi_{01} < \xi_{02} < \cdots$. The values of ξ_{0k} are available in Mathematica; here are the approximate values of the first several zeros: $\xi_{01} \approx 2.40483$, $\xi_{02} \approx 5.52008$, $\xi_{03} \approx 8.65373$, $\xi_{04} \approx 11.7915$, $\xi_{05} \approx 14.9309$, $\xi_{06} \approx 18.0711$, $\ldots$ (look at Figure 3).

(h) Write the functions $T_k(t)$ and the solution of the BVP for $v(s, t)$.

(i) Write the function $u(x, t)$ giving the position of the suspended string. (We have not imposed initial conditions, so that your expression will contain arbitrary constants.)