MATH 4073 **Homework 9** **Due Fri, 12/02/16**

Problem 1. Consider the function

$$f(t,y) = t^3 - 1 + t\cos^2 y \ .$$

Show that the function f satisfies a Lipschitz condition in the variable y and find its Lipschitz constant on the rectangle

$$D = [-5,1] \times \left[0, \tfrac{\pi}{6}\right] := \left\{(t,y) \in \mathbb{R}^2 : t \in [-5,1],\, y \in \left[0, \tfrac{\pi}{6}\right]\right\} \ .$$

Problem 2. The integral

$$M = \int_0^2 \mathrm{e}^{-x^2}\,\mathrm{d}x = 0.882081390762421679967481035914054037224051770868556468 01\ldots$$

cannot be expressed in terms of elementary functions, so it must be evaluated numerically.

(a) Find the maximum of the absolute value of the second derivative of the function $f(x) = \mathrm{e}^{-x^2}$ when x has an arbitrary value in the interval $[0,2]$.

(b) Use the formula for the error of the composite trapezoidal rule to give a rigorous upper bound on the absolute error in approximating the exact value M by the value coming from the composite trapezoidal rule by subdividing the interval $[0,2]$ into n equal subintervals.

(c) What is the smallest value of n you can take to ensure that the error does not exceed 10^{-6}?

(d) Compute the approximate value of M that you obtain by running the MATLAB code `comp_trap.m` (from the class web-site) with the value of n obtained in part (c). This code performs numerical integration by using the composite trapezoidal rule.

Hint: If you have forgotten how to run a MATLAB code, look at the instructions for running the code `bisection.m` in the materials accompanying Homework 2.

(e) Compute the actual value of the numerical error from part (d) and compare it with the admissible value of the error.

Problem 3.

(a) Show that the function $y(t)$ defined implicitly by the equation

$$y(t)^3 - t\sin\big(y(t)\big) + t^2 - 1 = 0 \tag{1}$$

satisfies the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\sin y - 2t}{3y^2 - t\cos y} \ , \qquad t \geq 0 \ , \\ y(0) &= 1 \ . \end{aligned} \tag{2}$$

(b) Find the numerical value of $y(\frac{1}{2})$ by solving the equation (1) using Newton's method. In other words, derive Newton's functional iteration and apply it to equation (1) by using the MATLAB code `newton.m` available on the class web-site. Use some reasonable value of the tolerance, say, 10^{-12} (recall that the accuracy of MATLAB is about 10^{-16}). Attach the MATLAB output of running the code `newton.m` in verbose mode.

(c) In the materials accompanying this homework on the class web-site, you will find the MATLAB codes `euler.m` and `rhs.m` needed in this part of the problem, as well as instructions how to use them.

Use the MATLAB code `euler.m` to solve the IVP (2) and find $y(\frac{1}{2})$. Do it with $N = 10$, 100, 1000, 10000, and 100000 (which corresponds to stepsize $h = 0.05$, 0.005, 0.0005, 0.00005 and 0.000005, respectively). In a table put the values of N, the corresponding values of $y(\frac{1}{2})_{\text{approx}}$ obtained by running `euler.m`, as well as the absolute errors $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, where $y(\frac{1}{2})_{\text{exact}}$ is the value found in part (a) by using Newton's method (using small enough tolerance, i.e., 10^{-12}).

(d) Plot by hand or using some software the logarithm of the error, $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the approximate straight line that goes through these points. How does the value of this slope match with the theoretical prediction for the value of the error of Euler's method?

Note that you can use natural logarithms or logarithms base 10, or any other base to plot the results (but use the same base for both axes!) – this is not going to change the slope of the approximate straight line.

Problem 4. The so-called *error function* is defined as

$$\operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t \mathrm{e}^{-x^2}\, \mathrm{d}x \ .$$

In Problem 2 you computed the value of the integral needed to find erf(2). In this problem you will find erf(2) by using a different method.

(a) Write an IVP for the function $\operatorname{erf}(t)$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \ , \qquad t \in [a, b] \ ,$$
$$y(a) = \alpha \ .$$

In other words, find the numerical values of a, b, and α, and the function $f(t, y)$.

(b) Use Euler's method with $N = 10$, 100, 1000, 10000, and 100000, to find erf(2).

Compare your results with the exact value, which is

$$\operatorname{erf}(2)_{\text{exact}} = 0.99532226501895273416206925636725292861089179704006007673 8\ldots \ .$$