MATH 4073 Homework 9 Due Fri, 11/19/10

Problem 1. Taras defined a family of polynomials which he (very modestly) denoted by T_0, T_1, T_2, ... These polynomials satisfy the following conditions:

(i) the polynomial T_k is of degree k;

(ii) $T_0(x) = 1$, and for all $k = 1, 2, 3, \ldots$, the coefficient of x^k in T_k is equal to 2^{k-1};

(iii) the polynomials T_0, T_1, T_2, ..., T_n form an orthogonal basis in the space of polynomials $V_n\left(-1, 1;\, w(x) = \frac{1}{\sqrt{1-x^2}}\right)$.

Recall that $V_n(a, b; w(x))$ stands for the linear space of polynomials of degree $\le n$ endowed with the inner product

$$(P, Q) = \int_a^b P(x)\, Q(x)\, w(x)\, \mathrm{d}x \ .$$

In the solution of this problem the following identities will be handy:

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx &= \pi \ , \\
\int_{-1}^{1} \frac{x^{2m}}{\sqrt{1-x^2}}\, dx &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)}{2^m m!}\, \pi \qquad \text{for } m = 1, 2, 3, \ldots \ , \\
\int_{-1}^{1} \frac{x^{2m-1}}{\sqrt{1-x^2}}\, dx &= 0 \ , \qquad \text{for } m = 1, 2, 3, \ldots \text{ (obviously!)} \ .
\end{aligned}$$

(a) Find the only polynomial T_1 of degree 1 of the form $T_1(x) = x + \cdots$ that is orthogonal to T_0 (recall that $T_0(x) = 1$ by definition).

(b) Find the only quadratic polynomial T_2 of the form $T_2(x) = 2x^2 + \cdots$ that is orthogonal to both T_0 and T_1.

(c) Show by direct integration that $(T_0, T_0) = \pi$, $(T_k, T_k) = \frac{\pi}{2}$ for $k = 1, 2$.

(d) Let $\tilde{T}_k := \mu_k T_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\left\|\tilde{T}_k\right\| := \sqrt{\left(\tilde{T}_k, \tilde{T}_k\right)} \ ,$$

of the polynomial $\tilde{T}_k$ is 1, for $k = 0, 1, 2, 3, \ldots$, and, therefore, the polynomials $\tilde{T}_0$, $\tilde{T}_1$, ..., $\tilde{T}_n$ form a *orthonormal* basis of the space $V_2\left(-1, 1; \frac{1}{\sqrt{1-x^2}}\right)$. Find the explicit expressions for $\tilde{T}_0(x)$, $\tilde{T}_1(x)$, and $\tilde{T}_2(x)$.

(e) Show that the polynomial $P(x) = 6x^2 - 5x + 4$ can be represented as a linear combination of the polynomials T_0, T_1 and T_2 as follows: $P = 3T_2 - 5T_1 + 7T_0$.

(f) What is the representation of P as a linear combination of the polynomials $\tilde{T}_0$, $\tilde{T}_1$ and $\tilde{T}_2$ found in part (d)?

(g) Find the orthogonal projection, $\mathrm{proj}_{T_0+2T_1} P$, of the polynomial $P(x) = 6x^2 - 5x + 4$ onto the " straight line"

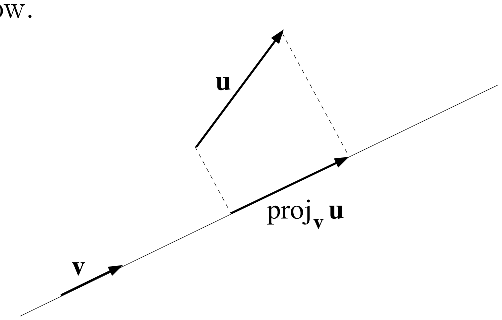
$$\ell := \{t(T_0 + 2T_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2\left(-1, 1; \frac{1}{\sqrt{1-x^2}}\right)$. If you have solved the previous parts of this problem, finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v})}{(\mathbf{v}, \mathbf{v})}\,\mathbf{v}$$

as in the picture below.



(h) Find the general form of all polynomials that are orthogonal to the polynomial $P = 3T_2 - 5T_1 + 7T_0$ in the sense of the inner product used in this problem. In more geometric terminology, find the subspace of the inner product linear space $V_2\left(-1, 1; \frac{1}{\sqrt{1-x^2}}\right)$ that is orthogonal to P.

Problem 2. This problem is an application of Taras's polynomials T_k studied in Problem 1 to Gaussian quadrature. The integrals given there will be useful for this problem as well. In this problem you will use the fact that the polynomials

$$T_0(x) = 1 \ , \quad T_1(x) = x \ , \quad T_2(x) = 2x^2 - 1 \ , \quad T_3(x) = 4x^3 - 3x$$

form an orthogonal basis in the inner product linear space $V_3\left(-1, 1; w(x) = \frac{1}{\sqrt{1-x^2}}\right)$ of all polynomials of degree ≤ 3 endowed with the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$. The goal of this problem is to find a Gaussian quadrature formula with degree of precision 5 based on the general formalism developed in class. The notations used are the same as in the handout "Theoretical foundations of Gaussian quadrature". Because of the specific form of the weight function, the formula you will develop will be particularly suitable for computing integrals of the form $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$.

In all parts of this problem, you need only to compute the quantities that are not given (e.g., in part (a) you only need to find x_3, in part (b) only $L_3(x)$, in part (c) only w_3). *Please show your calculations in detail!*

(a) Find the roots $x_1 < x_2 < x_3$ of the polynomial T_3.

Hint: I found that $x_1 = -\frac{\sqrt{3}}{2}$ and $x_2 = 0$.

(b) Write down the polynomials L_1, L_2, L_3.

Hint: I obtained the following for L_1 and L_2: $L_1(x) = \frac{2}{3}x^2 - \frac{1}{\sqrt{3}}x, \;\; L_2(x) = 1 - \frac{4}{3}x^2$.

(c) Find the weights w_1, w_2, w_3.

Hint: I computed that $w_1 = \frac{\pi}{3}$, $w_2 = \frac{\pi}{3}$.

(d) Write down the quadrature formula coming from parts (a), (b), (c).

(e) Show that the quadrature formula obtained in (d) is *exact* for all polynomials of degree 5.

(f) Show that the quadrature formula obtained in (d) is *not* exact for the polynomial $f(x) = x^6$. Does this agree with the theoretical prediction about the degree of precision of the method you developed?

(g) Now apply the beautiful quadrature formula you derived in (d) to compute the approximate value of the integral

$$\int_{-1}^{1} \frac{\mathrm{d}x}{(2-x)\sqrt{1-x^2}} \,,$$

whose exact value is $\frac{\pi}{\sqrt{3}}$. Find the numerical values of the absolute and the relative error.

Problem 3. One can show (but it is not easy!) that

$$I := \int_0^\infty \mathrm{e}^{-z^2}\,\mathrm{d}z = \frac{\sqrt{\pi}}{2} = 0.886226925452758\ldots\,.$$

We want to find the approximate value of this integral numerically, by using the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{3} w_i\, f(x_i) \,,$$

i	x_i	w_i
1	$-\sqrt{3/5} \approx -0.77459666924$	5/9
2	0	8/9
3	$\sqrt{3/5} \approx 0.77459666924$	5/9

Table 1: Nodes and weights of the Gaussian quadrature formula.

which is exact for polynomials of degree up to 5. The nodes x_i and the weights w_i are obtained from the corresponding Legendre polynomials, and can be shown to have the values given in Table 1 (you do *not* have to derive them).

(a) Define the functions φ_j $(j = 1, 2, 3)$ as follows:

$$\begin{aligned}
\varphi_1 &: (0, \infty) \to \left(0, \tfrac{\pi}{2}\right) : \xi \mapsto \varphi_1(\xi) = \arctan \xi \ , \\
\varphi_2 &: \left(0, \tfrac{\pi}{2}\right) \to (0, 2) : \xi \mapsto \varphi_2(\xi) = \frac{4}{\pi}\xi \ , \\
\varphi_3 &: (0, 2) \to (-1, 1) : \xi \mapsto \varphi_3(\xi) = \xi - 1 \ .
\end{aligned}$$

It is easy to see that these functions are invertible. Let φ be the composition of these three functions:

$$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1 \ .$$

Write this composition and its inverse explicitly:

$$\eta = \varphi(\xi) \ , \qquad \xi = \varphi^{-1}(\eta) \ .$$

What are the domains and the ranges of the functions φ and φ^{-1}?

(b) Use the functions constructed in part (a) to change variables in the integral $I = \int_0^\infty \mathrm{e}^{-z^2}\, \mathrm{d}z$ in order to transform it into an integral over the interval $[-1, 1]$. Write down the change of variables that does this, and the explicit expression of the integral in the new variables.

(c) Apply the quadrature formula based on the values of the nodes and weights from the table above to the integral constructed in part (b). What are the absolute and the relative error of your computation?

Remark: The relative error will be quite large (more than 7%), which should not be too surprising, because we are using a low accuracy Gaussian quadrature formula, and because we have transformed half of the real line into a finite interval.