

**Problem 1.** As we mentioned in class, one can define different norms in the same linear space. In this problem you will study different norms in  $\mathbb{R}^2$ . Let  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1\mathbf{i} + u_2\mathbf{j} \in \mathbb{R}^2$ .

- Define the norm  $\|\mathbf{u}\|_2$  by  $\|\mathbf{u}\|_2 := \sqrt{u_1^2 + u_2^2}$ . Draw the unit disk in  $\mathbb{R}^2$  in this norm, i.e., the set of vectors  $\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_2 \leq 1\}$ .
- Define the norm  $\|\mathbf{u}\|_1$  by  $\|\mathbf{u}\|_1 := |u_1| + |u_2|$ . Draw the unit disk in  $\mathbb{R}^2$  in this norm, i.e., the set of vectors  $\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_1 \leq 1\}$ .
- Define the norm  $\|\mathbf{u}\|_\infty$  by  $\|\mathbf{u}\|_\infty := \max\{|u_1|, |u_2|\}$ . Draw the unit disk in  $\mathbb{R}^2$  in this norm, i.e., the set of vectors  $\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_\infty \leq 1\}$ .
- Two norms  $\|\cdot\|$  and  $\|\cdot\|'$  on the same linear space are said to be *equivalent* if there exist positive constants  $C_1$  and  $C_2$  such that  $C_1\|\mathbf{u}\| \leq \|\mathbf{u}\|' \leq C_2\|\mathbf{u}\|$  for any vector  $\mathbf{u} \in V$ . Here we will prove that the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  on  $\mathbb{R}^2$  are equivalent:

$$\|\mathbf{u}\|_\infty = \max\{|u_1|, |u_2|\} \leq \sqrt{|u_1|^2 + |u_2|^2} = \|\mathbf{u}\|_2 ,$$

and

$$\|\mathbf{u}\|_2 = \sqrt{|u_1|^2 + |u_2|^2} \leq \sqrt{2 \max\{|u_1|^2, |u_2|^2\}} = \sqrt{2} \max\{|u_1|, |u_2|\} = \sqrt{2} \|\mathbf{u}\|_\infty .$$

The inequalities  $\|\mathbf{u}\|_\infty \leq \|\mathbf{u}\|_2 \leq \sqrt{2} \|\mathbf{u}\|_\infty$ , mean that the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are equivalent (the values of the constants are  $C_1 = 1$  and  $C_2 = \sqrt{2}$ ).

Show that the norms  $\|\mathbf{u}\|_1$  and  $\|\mathbf{u}\|_\infty$  are equivalent (you have to find the corresponding constants  $\tilde{C}_1$  and  $\tilde{C}_2$  such that  $\tilde{C}_1\|\mathbf{u}\|_1 \leq \|\mathbf{u}\|_\infty \leq \tilde{C}_2\|\mathbf{u}\|_1$ ).

- Use the fact that the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are equivalent and the fact (proved in part (d)) that the norms  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are equivalent to prove that the norms  $\|\mathbf{u}\|_1$  and  $\|\mathbf{u}\|_2$  are equivalent. In other words, you have to find constants  $C'_1$  and  $C'_2$  such that  $C'_1\|\mathbf{u}\|_1 \leq \|\mathbf{u}\|_2 \leq C'_2\|\mathbf{u}\|_1$ . This won't require any additional calculations – simply express the constants  $C'_1$  and  $C'_2$  in terms of  $C_1$ ,  $C_2$ ,  $\tilde{C}_1$ , and  $\tilde{C}_2$ .

**Problem 2.** Let  $V_n(a, b; w(x))$  stand for the linear space of polynomials of degree no greater than  $n$  endowed with the inner product

$$\langle P, Q \rangle = \int_a^b P(x) Q(x) w(x) dx .$$

We want to construct polynomials  $P_0, P_1, \dots, P_n$  satisfying the following conditions:

- (i) the polynomial  $P_k$  is of degree  $k$ ;
- (ii) then coefficient of  $x^k$  in  $P_k$  is equal to 1 (such polynomials are called *monic*);
- (iii) the polynomials  $P_0, P_1, P_2, \dots, P_n$  form an orthogonal basis in the space of polynomials  $V_n(0, \infty; w(x) = e^{-x})$ .

In the solution of this problem the following identity will be handy:

$$\int_0^{\infty} x^k e^{-x} dx = k!$$

(where, by definition,  $0! = 1$ ).

- (a) Clearly,  $P_0(x) = 1$  for each  $x \in [0, \infty)$ . Find the only monic polynomial  $P_1$  of degree 1 that is orthogonal to  $P_0$ . Clearly,  $P_1$  should have the form  $P_1(x) = x + \alpha$ , where  $\alpha$  is a constant whose value you have to find. (The coefficient multiplying  $x$  is 1 because we want the polynomials  $P_k$  to be monic.)
- (b) Find the only monic quadratic polynomial  $P_2$  that is orthogonal to both  $P_0$  and  $P_1$ . The polynomial  $P_2$  should have the form  $P_2(x) = x^2 + \beta x + \gamma$ , where  $\beta$  and  $\gamma$  are constants whose values you have to find. (*Hint*: I obtained that  $\gamma = 2$ .)
- (c) Show that the polynomial  $Q(x) = x^2 + 3$  can be represented as a linear combination of the polynomials  $P_0, P_1$  and  $P_2$  as follows:  $Q = P_2 + 4P_1 + 5P_0$ .
- (d) Show by direct integration that  $\langle P_0, P_0 \rangle = 1$ ,  $\langle P_1, P_1 \rangle = 1$ ,  $\langle P_2, P_2 \rangle = 4$ .
- (e) Find the orthogonal projection,  $\text{proj}_{P_0+2P_1} Q$ , of the polynomial  $Q(x) = x^2 + 3$  onto the “straight line”

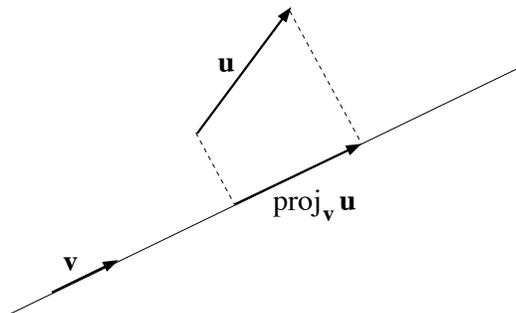
$$\ell := \{t(P_0 + 2P_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space  $V_2(0, \infty; e^{-x})$ . If you have solved part (c), then finding this orthogonal projection should be easy.

*Hint*: If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in the inner product linear space  $V$ , then the orthogonal projection of the vector  $\mathbf{u}$  onto the straight line in the direction of  $\mathbf{v}$  is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

– see the picture below.



(f) Finally, let  $\tilde{P}_k := \mu_k P_k$ , where  $\mu_k > 0$  is a constant (depending on  $k$ ) such that the norm,

$$\|\tilde{P}_k\| := \sqrt{\langle \tilde{P}_k, \tilde{P}_k \rangle},$$

of the polynomial  $\tilde{P}_k$  is 1. Find the explicit expressions for  $\tilde{P}_0(x)$ ,  $\tilde{P}_1(x)$ , and  $\tilde{P}_2(x)$ .

**Problem 3.** As you know, one way to approximate a function  $f$  of one variable is to replace it by its tangent line at some point of interest, or by the “best fitting” parabola at this point (these approximations correspond to using the first- or second-order Taylor polynomial of the function  $f$  at this point). This type of approximation, however, works very well only near this point, and can be very inaccurate over an entire *interval*.

One way to approximate a function  $f$  (of one variable) on an entire interval is the following. Choose some class of functions  $\mathcal{H}$ , say all linear functions. Then look for a function  $h$  from this class  $\mathcal{H}$  for which the “distance” between  $f$  and  $h$  is the smallest possible. The “distance” – which is usually called “error” – can be defined in many different ways. If we want to approximate  $f$  by a function  $h \in \mathcal{H}$  on the interval  $[a, b]$ , and we want  $|f(x) - h(x)|$  to be small for all  $x \in [a, b]$ , then an appropriate definition for the “error” would be  $E_\infty := \max_{x \in [a, b]} |f(x) - h(x)|$ . Another choice is to minimize  $E_1 := \int_a^b |f(x) - h(x)| dx$ , but the expressions for  $E_\infty$  and  $E_1$  cause technical difficulties if one tries to use them in practice. The most convenient for numerical purposes expression for the error is

$$E_2 := \int_a^b [f(x) - h(x)]^2 dx,$$

which we will use below. Incidentally, the cryptic notations  $E_\infty$ ,  $E_1$ , and  $E_2$  are similar to the notations for the norms  $\|\cdot\|_\infty$ ,  $\|\cdot\|_1$ , and  $\|\cdot\|_2$ .

In this problem you will find the best approximation of the function  $f(x) = x^3$  by a linear function,  $h_{\mu, \nu}(x) := \mu x + \nu$ , over the interval  $[0, 1]$  if the “error” is given by the integral

$$E_f(\mu, \nu) := \int_0^1 [f(x) - h_{\mu, \nu}(x)]^2 dx. \quad (1)$$

In other words, you have to choose the values of the constants  $\mu$  and  $\nu$  that minimize the error  $E_f(\mu, \nu)$  given by (1).

*Hint:* Here is a useful fact:  $\int_0^1 [x^3 - (\mu x + \nu)]^2 dx = \frac{1}{7} - \frac{2}{5} \mu + \frac{1}{3} \mu^2 - \frac{1}{2} \nu + \mu \nu + \nu^2$ .

**Problem 4.** In this problem you will solve in a geometric way the problem of finding the linear function that is “closest” to the function  $x^3$  in the sense that it minimizes the “error” (1). Let  $V_3(0, 1)$  stand for the linear space of polynomials on the interval  $[0, 1]$  of degree no greater than 3, endowed with the inner product

$$\langle P, Q \rangle = \int_0^1 P(x) Q(x) dx, \quad P \in V_3(0, 1), \quad Q \in V_3(0, 1).$$

Below we will use the “quantum mechanical notation”

$$\langle P|Q\rangle := \langle P, Q\rangle ,$$

where  $\langle P|$  is the “bra-vector” corresponding to the “ket-vector”  $|P\rangle$ .

It is easy to show directly that the polynomials

$$N_0(x) = 1$$

$$N_1(x) = x - \frac{1}{2}$$

$$N_2(x) = x^2 - x + \frac{1}{6}$$

$$N_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

form an orthogonal basis of the space  $V_3(0, 1)$ ; the norms of these vectors are

$$\|N_0\| = \sqrt{\langle N_0, N_0\rangle} = 1 , \quad \|N_1\| = \frac{1}{\sqrt{12}} , \quad \|N_2\| = \frac{1}{\sqrt{180}} , \quad \|N_3\| = \frac{1}{\sqrt{2800}} ;$$

you do *not* need to do any of these calculations. This basis has the property that  $N_k$  is a polynomial of degree  $k$ .

Let  $Q \in V_3(0, 1)$  be the polynomial

$$Q(x) = x^3 .$$

As in Problem 3, we want to find a linear function, i.e., a polynomial of degree no more than 1 that is “closest” to  $Q$ ; such polynomials form a subspace of  $V_3(0, 1)$  which we will denote by  $V_1(0, 1)$ :

$$V_1(0, 1) = \{ L : [0, 1] \rightarrow \mathbb{R} \mid L(x) = \mu x + \nu, \mu \in \mathbb{R}, \nu \in \mathbb{R} \} .$$

Since  $N_k$  is a polynomial of degree  $k$ , any polynomial of degree 1 – i.e., every  $L \in V_1(0, 1)$  – is a linear combination of  $N_0$  and  $N_1$ , so that we can write

$$V_1(0, 1) = \text{span} \{ N_0, N_1 \} = \{ L = \alpha N_0 + \beta N_1 \mid \alpha \in \mathbb{R}, \beta \in \mathbb{R} \} . \quad (2)$$

Recall that, for any vector  $P \in V_3(0, 1)$ , the operator

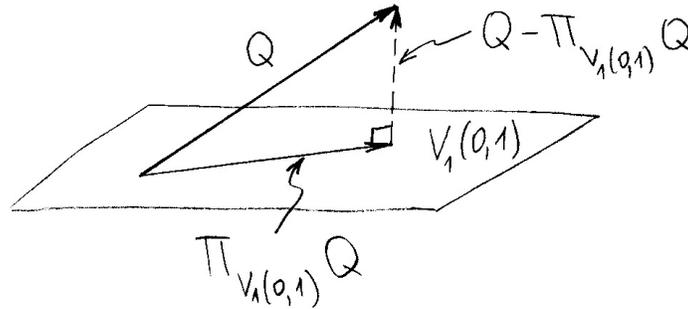
$$\Pi_P := \frac{|P\rangle\langle P|}{\|P\|^2}$$

is the orthogonal projection of an arbitrary vector  $Q \in V_3(0, 1)$  onto the direction of  $P$ :

$$\Pi_P|Q\rangle = \frac{|P\rangle\langle P|}{\|P\|^2} |Q\rangle = \frac{|P\rangle\langle P|Q\rangle}{\|P\|^2} .$$

Similarly, since the vectors  $N_0$  and  $N_1$  are orthogonal to one another, the orthogonal projection onto the plane  $V_1(0, 1) = \text{span}\{N_0, N_1\}$  is given by the operator

$$\Pi_{V_1(0,1)} := \Pi_{N_0} + \Pi_{N_1} = \frac{|N_0\rangle\langle N_0|}{\|N_0\|^2} + \frac{|N_1\rangle\langle N_1|}{\|N_1\|^2}.$$



The projection of an arbitrary vector  $|Q\rangle \in V_3(0, 1)$  onto the plane  $V_1(0, 1)$  is, therefore, given by  $\Pi_{V_1(0,1)}|Q\rangle \in V_1(0, 1)$ . It can be shown that, among all vectors in  $V_1(0, 1)$ , the vector  $\Pi_{V_1(0,1)}|Q\rangle$  is the one that is “closest” to  $|Q\rangle \in V_3(0, 1)$  in the sense, that norm of the difference

$$|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle$$

is the smallest. Note that, if  $|Q\rangle \in V_1(0, 1)$ , then  $|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle = 0$ .

- (a) Check that the vector  $Q \in V_3(0, 1)$  defined by  $Q(x) = x^3$  can be written as a linear combination of the vectors from the orthogonal basis  $\{N_0, N_1, N_2, N_3\}$  of  $V_3(0, 1)$  as

$$Q = \frac{1}{4}N_0 + \frac{9}{10}N_1 + \frac{3}{2}N_2 + N_3$$

(i.e.,  $x^3 = \frac{1}{4}N_0(x) + \frac{9}{10}N_1(x) + \frac{3}{2}N_2(x) + N_3(x)$ ).

- (b) Show that the projection  $\Pi_{V_1(0,1)}|Q\rangle$  is equal to  $\frac{1}{4}N_0 + \frac{9}{10}N_1$ .

*Hint:* This is very easy if you use the result of part (a) and the fact that the basis  $\{N_0, N_1, N_2, N_3\}$  of  $V_3(0, 1)$  is orthogonal.

- (c) Compare your answer to part (b) with the function  $h_{\mu,\nu}$  that you found in Problem 3.

**Problem 5.** In this problem you will answer in a different way the same question as in Problems 3 and 4.

Looking at the figure in Problem 4, we see that, the shortest distance from “the end” of the vector  $Q$  to the plane  $V_1(0, 1)$  is accomplished if the difference  $|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle$  is perpendicular to the plane  $V_1(0, 1)$ . Since  $\Pi_{V_1(0,1)}|Q\rangle$  belongs to  $V_1(0, 1)$  which was defined

(recall (2)) as the span of the vectors  $N_0$  and  $N_1$ , i.e., the set of all linear combinations of  $N_0$  and  $N_1$ . Therefore, we have

$$\Pi_{V_1(0,1)}|Q\rangle = a|N_0\rangle + b|N_1\rangle .$$

Therefore, the vector

$$|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle = |Q\rangle - (a|N_0\rangle + b|N_1\rangle) = |Q\rangle - a|N_0\rangle - b|N_1\rangle \quad (3)$$

must be orthogonal to any vector from  $V_1(0,1)$ , which is equivalent to saying that it is orthogonal to each of the vectors  $|N_0\rangle$  and  $|N_1\rangle$  that “generate” the plane  $V_1(0,1)$ .

(a) Write down the conditions

$$(|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle) \perp N_0 ,$$

$$(|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle) \perp N_1$$

for  $Q(x) = x^3$  and  $N_0$  and  $N_1$  given in Problem 4, and derive a system of two equations for the constants  $a$  and  $b$  in (3). Your calculations will be greatly simplified if you use the representation of  $Q$  as a superposition of the vectors  $N_j$  that was derived in part (a) of Problem 4.

(b) Solve the system obtained in part (a). Compare your result for the vector  $a|N_0\rangle + b|N_1\rangle \in V_1(0,1)$  from (3) with the vector  $h_{\mu,\nu}$  obtained in Problem 3 and the vector  $\Pi_{V_1(0,1)}|Q\rangle$  obtained in Problem 4.