MATH 5763 Homework 8 Due Thu, 4/11/2019

Problem 1. [Renewal equation]
Let X_1, X_2, $\ldots$ be a sequence of independent Uniform$(0,1)$ random variables, $t \in (0,1]$, and

$$N(t) := \min\left\{ n \in \mathbb{N} \; : \; \sum_{k=1}^{n} X_k > t \right\}$$

be the smallest number of X's that need to be added so that their sum exceeds t, as shown in Figure 1 (where $N(t) = 4$). Define $M(t) := \mathbb{E}[N(t)]$. From the definition, it is clear that $N(t)$ is a random variable taking values in $\mathbb{N}$.

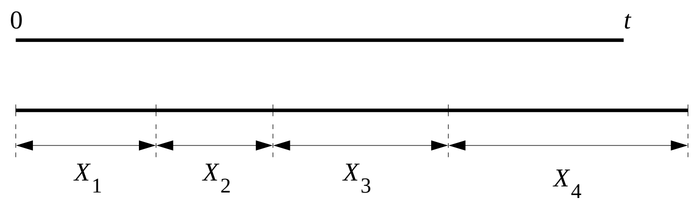

Figure 1: Covering the interval $[0,t]$ with lines with random lengths X_m.

(a) Explain why, for any $x_1 \in (0,1]$,

$$\mathbb{E}\big[N(t)\big|X_1 = x_1\big] = \begin{cases} 1 & \text{for } t < x_1 \; , \\ 1 + M(t - x_1) & \text{for } x_1 \leq t \; . \end{cases}$$

Hint: Condition on X_1. Please explain your reasoning (no calculations are needed)!

(b) Show that $M(t)$ satisfies the integral equation

$$M(t) = 1 + \int_0^t M(t - x_1)\,\mathrm{d}x_1 \; . \tag{1}$$

Hint: Use the Tower Rule by conditioning on X_1. Do not forget that $t \in (0,1]$.

(c) Solve the integral equation for $M(t)$ by first converting it to a differential equation. First change variables in the integral to rewrite (1) in the form $M(t) = 1 + \int_0^t M(y)\,\mathrm{d}y$ (write explicitly the change of variables). Differentiate both sides of this equation to derive a *differential* equation for $M(t)$. What is the initial condition for $M(0)$ that the function $M(t)$ should satisfy (justify your answer with a couple of sentences). Solve the initial value problem for $M(t)$ you have obtained.

(d) Now solve the integral equation for $M(t)$ by using Laplace transform. You can use the following facts about Laplace transform, without proving them:

– if $h(x) = x^n$, for $n \in \mathbb{N}$, then $\mathcal{L}[h](\xi) = \dfrac{n!}{\xi^{n+1}}$, $\xi > 0$;

– if $h(x) = \mathrm{e}^{ax}$ (for any $a \in \mathbb{R}$), then $\mathcal{L}[h](\xi) = \dfrac{1}{\xi - a}$, $\xi > a$;

– if $A(t) = \displaystyle\int_0^t M(t - x_1)\, f(x_1)\, \mathrm{d}x_1$, then $\mathcal{L}[A](\xi) = \mathcal{L}[M](\xi)\, \mathcal{L}[f](\xi)$.

Problem 2. [Autocorrelation and autocovariance functions of a Poisson process] Let $N = \{N_t : t \geq 0\}$ be a Poisson process with a constant rate λ.

(a) Compute the expected value $m_N(t) = \mathbb{E}[N_t]$.

Hint: Use that N_t is a Poisson distributed random variable with parameter λt. Recall that if $X \sim \text{Poisson}\,(\alpha)$, then $\mathbb{E}[X] = \alpha$ and $\operatorname{Var} X = \alpha$.

(b) Use your result in (a) and the basic properties of Poisson processes to find the autocorrelation function, $R_N(s,t) = \mathbb{E}[N_s N_t]$, of the Poisson process N.

Hint: First assume that $0 \leq s \leq t$ and compute $R_N(s,t)$; then think how your result will change if $0 \leq t \leq s$, and write your result for $R_N(s,t)$ in the general case.

(c) Compute the autocovariance function,

$$C_N(s,t) = \mathbb{E}[(N_s - \mathbb{E}N_s)(N_t - \mathbb{E}N_t)] = R_N(s,t) - m_N(s)m_N(t)\ .$$

Look at $C_N(t,t)$ and at the hint to part (a) – are the results for $C_N(t,t)$ and $\operatorname{Var} X_t$ consistent?

(d) Compute the correlation coefficient, $\rho_N(s,t) = \dfrac{C_N(s,t)}{\sqrt{C_N(s,s)\, C_N(t,t)}}$. From your result, find $\rho_N(s,t)$ for $t \geq s$ and find $\displaystyle\lim_{t \to s^+} \rho_N(s,t)$. Is this what you expected? Why?

(e) Is N a wide-sense stationary process? Explain briefly.

(f) Is N a strong-sense stationary process? Why?

Problem 3. [P.m.f. of order k and conditional transition function of a Markov process] Let $X = \{X_t : t \geq 0\}$ be a continuous-time Markovian process with state space $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$. Let $N = \{N_t : t \geq 0\}$ be a Poisson process with a constant rate λ.

Recall that the *probability mass function (p.m.f.) of order k* for the process X is defined as the joint p.m.f. of the random vector $(X_{t_1}, \ldots, X_{t_k})$

$$p^{(k)}(i_1, \ldots, i_k\, ;\, t_1, \ldots, t_k) := p_{(X_{t_1}, \ldots, X_{t_k})}(i_1, \ldots, i_k) = \mathbb{P}(X_{t_1} = i_1, \ldots, X_{t_k} = i_k)\ ,$$

where $t_m \geq 0$ and $i_m \in \mathbb{Z}_+$ for $m = 1, \ldots, k$ (see Definiiton 2.1.4 on page 49 of Lefebvre's book, where the notation is slightly different).

The *conditional transition function* of the process X is defined as the conditional p.m.f. of X at a moment t conditioned on an earlier moment t_0:

$$p(i, i_0\,;\, t, t_0) := p_{X_t|X_{t_0}}(i|i_0) = \mathbb{P}(X_t = i|X_{t_0} = i_0) = \frac{\mathbb{P}(X_t = i, X_{t_0} = i_0)}{\mathbb{P}(X_{t_0} = i_0)} \ , \qquad t_0 < t$$

(compare this with its continuous analogue, defined in Eqn. (2.49) on page 62 of Lefebvre's book). Throughout this problem, t_0, t_1, and t_2 are times satisfying

$$0 \le t_0 \le t_1 \le t_2 \ .$$

(a) Use some basic property of probability to explain why

$$\sum_{i_1} \mathbb{P}(X_{t_2} = i_2, X_{t_1} = i_1, X_{t_0} = i_0) = \mathbb{P}(X_{t_2} = i_2, X_{t_0} = i_0) \ .$$

(b) Use your result from part (a) to show that the conditional transition function of the process X satisfies

$$\sum_{i_1} p(i_2, i_1\,;\, t_2, t_1)\, p(i_1, i_0\,;\, t_1, t_0) = p(i_2, i_0\,;\, t_2, t_0) \ .$$

This equation has a name – what is it?

Hint: Write

$$\sum_{i_1} p(i_2, i_1\,;\, t_2, t_1)\, p(i_1, i_0\,;\, t_1, t_0) = \mathbb{P}(X_{t_2} = i_2 \,|\, X_{t_1} = i_1)\, \mathbb{P}(X_{t_1} = i_1 \,|\, X_{t_0} = i_0) \ ,$$

then explain why $\mathbb{P}(X_{t_2} = i_2 \,|\, X_{t_1} = i_1) = \mathbb{P}(X_{t_2} = i_2 \,|\, X_{t_1} = i_1, X_{t_0} = i_0)$ (it relies on one property of the process X – which one?), and use the definition of conditional probability to derive the desired equality. The calculations are really simple (look for a nice cancellation).

(c) Prove that

$$\sum_{i_0} p(i_1, i_0\,;\, t_1, t_0)\, p^{(1)}(i_0\,;\, t_0) = p^{(1)}(i_1\,;\, t_1) \ .$$

(d) Explain in words the intuitive meaning of the identity

$$\sum_{i_0} p^{(2)}(i_0, i_1\,;\, t_0, t_1) = p^{(1)}(i_1\,;\, t_1) \ , \qquad t_0 < t_1 \ .$$

(e) Explain in words the intuitive meaning of the identity

$$\sum_{i_1} p^{(3)}(i_0, i_1, i_2\,;\, t_0, t_1, t_2) = p^{(2)}(i_0, i_2\,;\, t_0, t_2) \ , \quad t_0 < t_1 < t_2 \ .$$

(f) In the rest of the problem you will consider the particular case of the Poisson process N with rate $\lambda = \text{const}$. What is $p^{(1)}(i_0\,;\, t_0) = \mathbb{P}(N_{t_0} = i_0)$? You can write the answer very simply; use whatever you know about the Poisson process.

(g) What is $p^{(2)}(i_0, i_1\,;\, t_0, t_1)$ for $0 \le t_0 < t_1$? Clearly, since the Poisson process is non-decreasing, $p^{(2)}(i_0, i_1\,;\, t_0, t_1)$ will be non-zero only for $0 \le i_0 \le i_1$.

Hint: The independence of increments of the Poisson process will allow you to answer this question with practically no calculations.

(h) Use reasoning similar to part (g) to write $p^{(3)}(i_0, i_1, i_2\,;\, t_0, t_1, t_2)$ for $0 \le t_0 < t_1 < t_2$ for the Poisson process; the result will be non-zero only for $0 \le i_0 \le i_1 \le i_2$.

(i) Use the concrete expressions for $p^{(1)}(i_0\,;\, t_0)$ and $p^{(2)}(i_0, i_1\,;\, t_0, t_1)$ to prove the identity

$$\sum_{i_0} p^{(2)}(i_0, i_1\,;\, t_0, t_1) = p^{(1)}(i_1\,;\, t_1)\;, \qquad t_0 < t_1$$

for the particular case of a Poisson process N with constant rate λ. What should be the range for the summation index i_0, and why? You will need the binomial formula.

Problem 4. [Autocorrelation and autocovariance functions of a flip-flop process]

Consider again the flip-flop process $X = \{X_t \,:\, t \ge 0\}$ from Problem 3 of Homework 6. Namely, assume that $N = \{N_t \,:\, t \ge 0\}$ is a Poisson process with intensity λ, and define the flip-flop process $X = \{X_t \,:\, t \ge 0\}$ with state space $S = \{0, 1\}$ by

$$X_t = \frac{1}{2} + (-1)^{N_t}\left(X_0 - \frac{1}{2}\right)\;,$$

where X_0 is a random variable with values in S that is independent of the process N. This complicated formula simply means that the process X switches between 0 and 1 at each event of N. Since N is a time-homogeneous Markov chain, X is also a time-homogeneous Markov chain. Assume that the initial state X_0 of the Markov chain X is a random variable with distribution $p_j(0) = \mathbb{P}(X_0 = j) = \dfrac{1}{2}$ for $j = 0, 1$. Note that $\mathbf{p}(0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the stationary distribution of the chain X, as you found in Homework 6. Recall that in Homework 6 you have found (in several ways) the stochastic semigroup $\mathbf{P}_t$ of the continuous-time Markov chain X:

$$\mathbf{P}_t = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t}) \\ \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1 + \mathrm{e}^{-2\lambda t}) \end{pmatrix}\;.$$

(a) Show the distribution of the random variable X_t is $\mathbf{p}(t) = \begin{pmatrix} p_0(t) & p_1(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (where $p_i(t) = \mathbb{P}(X_t = i)$). You should do this in two ways: (1) by a calculation using the explicit expressions for $\mathbf{P}_t$ and $\mathbf{p}(0)$, and (2) by giving a simple reason why this is the answer, without any calculations.

(b) For $s \ge 0$, $t \ge 0$, show that the expectation of X_t is $m_X(t) := \mathbb{E}[X_t] = \frac{1}{2}$, and the autocorrelation function is $R_X(s, s+t) := \mathbb{E}[X_s X_{s+t}] = \frac{1}{4}\left(1 + \mathrm{e}^{-2\lambda t}\right)$.

Hint: To compute the expected value of the product $X_s X_{s+t}$, use the Tower Rule by conditioning on the state of the system at the earlier time, namely, $\mathbb{E}[X_s X_{s+t}] = \mathbb{E}[\mathbb{E}[X_s X_{s+t} | X_s]]$. You can first compute

$$\mathbb{E}[X_s X_{s+t} | X_s = j] = j\, \mathbb{E}[X_{s+t} | X_s = j] = j \sum_{k=0}^{1} k\, p_{jk}(t) = \cdots\;,$$

and then apply the Tower Rule,

$$\mathbb{E}[X_s X_{s+t}] = \mathbb{E}[\mathbb{E}[X_s X_{s+t}|X_s]] = \sum_{j=0}^{1} \mathbb{E}[X_s X_{s+t}|X_s = j]\,\mathbb{P}(X_s = j) = \cdots \ .$$

(c) Is the process X weakly stationary? Explain briefly.

(d) Compute the autocovariance function

$C_X(s, s+t) := \mathrm{Cov}\,(X_s, X_{s+t}) = \mathbb{E}[(X_s - \mathbb{E}X_s)(X_{s+t} - \mathbb{E}X_{s+t})] = R_X(s, s+t) - m_X(s)\, m_X(s+t)$,

and show that the autocorrelation function is equal to $\mathrm{e}^{-2\lambda t}$.

Food for Thought Problem 1. [Riemann-Stieltjes integral]

Let the function $F : \mathbb{R} \to [0, 1]$ shown in Figure 2 be defined by $F(x) = 0$ for $x \leq 0$, $F(x) = 1$ for $x \geq 1$, and

$$F(x) = \frac{1}{2^{j-1}} \qquad \text{for} \qquad \frac{1}{2^j} \leq x < \frac{1}{2^{j-1}} \ , \quad j \in \{1, 2, 3, \ldots\}$$

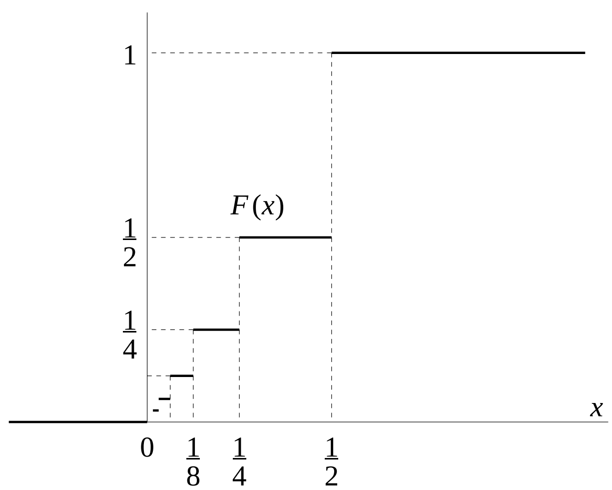

Figure 2: Graph of the function F.

Show that $\displaystyle\int_{\mathbb{R}} \mathrm{d}F(x) = 1$, $\displaystyle\int_{\mathbb{R}} x\,\mathrm{d}F(x) = \frac{1}{3}$, $\displaystyle\int_{\mathbb{R}} \ln x\,\mathrm{d}F(x) = -2\ln 2$.

Hint: You will need that, for $|q| < 1$, $\displaystyle\sum_{k=1}^{\infty} k\,q^{k-1} = \frac{\mathrm{d}}{\mathrm{d}q}\sum_{k=0}^{\infty} q^k = \frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{1-q} = \frac{1}{(1-q)^2}$.

Food for Thought Problem 2. [Delta function and its derivatives]

Use the definition of Dirac δ-function and its derivatives to show that

$$\int_{\mathbb{R}} x^2\,\delta_3(x)\,\mathrm{d}x = 9 \ , \quad \int_{\mathbb{R}} \mathrm{e}^{x^2}\,\delta_3'(x)\,\mathrm{d}x = -6\mathrm{e}^9 \ , \quad \int_{\mathbb{R}} \mathrm{e}^{x^2}\,\delta_3''(x)\,\mathrm{d}x = 38\mathrm{e}^9 \ .$$