MATH 5763 **Homework 8** **Due Thu, 11/12/2015**

Problem 1. Vehicles pass a pedestrian crossing at the instants of a Poisson process of intensity λ. It takes you a seconds to cross the street, so you need that the time interval between two successive cars passing by you is at least of length a. Let T be the first time at which you would succeed in crossing to the other side; note that T is the sum of the time that you may need to wait on the sidewalk before starting to cross the street and the time a during which you are crossing the street. The situation is shown pictorially in Figure 1.

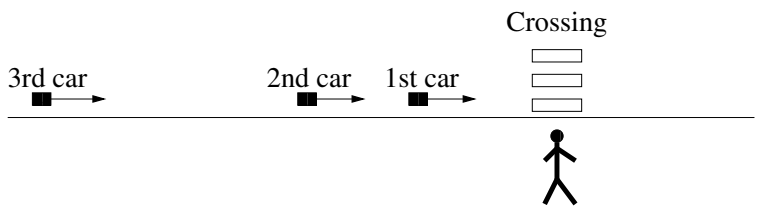

Figure 1: A pedestrian is waiting to cross the street.

In Figure 2, you see the time axis (assume that you started the stopwatch at the moment you arrived at the street), with the moments T_1, T_2, and T_3 of the first, second, and third cars passing the crossing, and the time interval a you need in order to cross. Clearly, a does not fit between 0 and T_1, as well as between T_1 and T_2; it, however, fits between T_2 and T_3, so you will start crossing at $t = T_2$, and reach the other side of the street at time $t = T_2 + a$, so that in this case the random variable T has value $T = T_2 + a$, as shown in the figure.

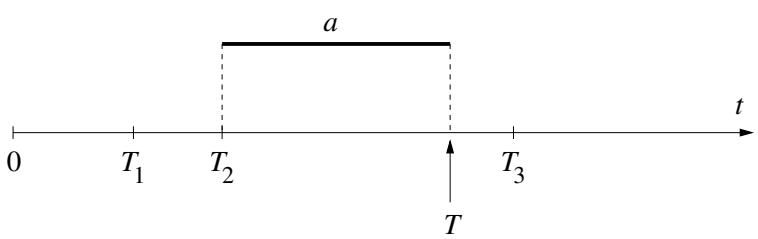

Figure 2: On the meaning of the random variable T.

(a) Explain in a couple of sentences why

$$\mathbb{E}[T|T_1 = t_1] = \begin{cases} a\ , & t_1 > a, \\ t_1 + \mathbb{E}[T]\ , & t_1 < a\ . \end{cases}$$

(b) Use your result from part (a) and the Tower Rule in the form $\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|T_1]]$ to show that the expectation $\mathbb{E}[T]$ satisfies the integral equation

$$\mathbb{E}[T] = \int_0^a (x + \mathbb{E}[T])\, \lambda \mathrm{e}^{-\lambda x}\, \mathrm{d}x + a\mathrm{e}^{-\lambda a}\ .$$

(c) Solve the integral equation for $\mathbb{E}[T]$ derived in part (b).

Hint: This is very easy, you don't need to apply any sophisticated techniques! For your convenience, here are some integrals:

$$\int_0^a x\lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x = \frac{1}{\lambda} - \frac{1}{\lambda}\mathrm{e}^{-\lambda a} - a\mathrm{e}^{-\lambda a}$$

$$\int_0^a \lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x = 1 - \mathrm{e}^{-\lambda a}\ .$$

(d) Does the expression for $\mathbb{E}[T]$ you derived in part (c) behave reasonably (i.e., matches your intuition) in the limits $\lambda \to 0$ and $\lambda \to \infty$? Discuss briefly for each of these two cases.

(e) Now suppose that there are two lanes to cross, carrying independent Poissonian traffic with respective intensities λ and μ. You need time a to cross the first lane and time b to cross the second lane. Find the expected total time to cross the street in the following two cases:

(i) there is an island between the two lanes where you can stay safely;

(ii) you must cross both lanes at once, for which, of course, you will need time $a + b$;

Hint: You can use your results from part (c) applied to cases (i) and (ii). Recall that the superposition of two Poisson processes is a Poisson process (see Problem 4(a) of Homework 5).

(f) Which of the total times found in (e) is greater – in case (i) or in case (ii)? Of course, you can answer this question by comparing your answers in the cases (i) and (ii) above, but don't do this here – just give a simple intuitive argument.

Problem 2. In class we considered an M(λ)/G/1 queueing process $Q(t)$ where the arrival of customers is according to Poisson process of intensity λ (i.e., the inter-arrival times are i.i.d. random variables of type Exp(λ)), the service times are independent random variables with general (known) distribution, and one server. Let the service times S_n (where S_n is the time it takes to serve the nth customer) be independent random variables (which are also independent of the arrival times) with a common distribution that is known (i.e., you assume that you know the p.m.f. p_S of S_n if it is a discrete random variable, or the p.d.f. f_S of S_n if it is continuous).

Since such processes are difficult to deal with since they are not Markov, we defined a discrete stochastic process embedded in the queueing process $Q(t)$ that turns out to be Markov. Let D_n be the time of departure of the nth customer from the system, and $Y_n := Q(D^+)$ be the number of customers which the nth customer leaves behind after he/she leaves the queue. To show that $\{Y_n\}_{n=0}^{\infty}$ is a Markov process, let U_n be the number of customers arriving during the service time (of length S_{n+1}) of the $(n+1)$st customer. In class we showed that

$$Y_{n+1} = \begin{cases} U_n & \text{if } Y_n = 0\ , \\ Y_n + U_n - 1 & \text{if } Y_n \geq 1\ . \end{cases}$$

This can also be written as

$$Y_{n+1} = Y_n + U_n - 1 + h(Y_n)\ , \qquad \text{where } h(Y_n) := \begin{cases} 1 & \text{if } Y_n = 0\ , \\ 0 & \text{if } Y_n \geq 1\ . \end{cases}$$

Since the process of arrival is Poisson of intensity λ, the number of customers arriving during time interval of duration S_{n+1} is $U_n \sim \text{Poisson}(\lambda S_{n+1})$, from where it is clear that the discrete-time process $\{Y_n\}$ embedded in the continuous-time queueing process $\{Q(t)\}$ is a Markov process. We derived the transition probability matrix $\mathbf{P}$ of this discrete-time process:

$$\mathbf{P} = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots \\ 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots \\ 0 & 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \cdots \\ 0 & 0 & 0 & \delta_0 & \delta_1 & \delta_2 & \cdots \end{pmatrix} , \qquad \text{where } \delta_j = \mathbb{E}\left[\frac{(\lambda S)^j}{j!}\,\mathrm{e}^{-\lambda S}\right] . \tag{1}$$

Define the *trafic intensity* (also called *traffic density*, or *offered load*) of the queue as

$$\rho := \frac{\text{expected service time}}{\text{expected interarrival time}} = \frac{\mathbb{E}[S]}{1/\lambda} = \lambda \mathbb{E}[S] .$$

One can show that for the embedded Markov chain $\{Y_n\}$ is positive recurrent if and only if $\rho < 1$; if $\rho = 1$, $\{Y_n\}$ is null-recurrent, and if $\rho > 1$, $\{Y_n\}$ is transient. If $\rho < 1$, $\{Y_n\}$ has a stationary distribution.

(a) Check that the matrix $\mathbf{P} = (p_{ij})_{i,j=0}^{\infty}$ defined by (1) is a stochastic matrix in general (i.e., for any distribution of the service time S).

(b) Compute δ_j in the case of $\mathrm{M}(\lambda)/\mathrm{M}(\mu)/1$ queue, i.e., when S is a random variable of type $\mathrm{Exp}(\mu)$ (in which case $\mathbb{E}[S] = \frac{1}{\mu}$).

Hint: Recall that $\Gamma(k) = \displaystyle\int_0^\infty \xi^{k-1}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi = (k-1)!$ for any $k \in \mathbb{N}$.

(c) Compute δ_j if S is deterministic, taking value $\frac{1}{\mu}$ with probability 1. (We take S to be equal to $\frac{1}{\mu}$ so that $\mathbb{E}[S]$ be the same as in part (b).)

(d) Compute δ_j if S is uniformly distributed on $[0, \frac{2}{\mu}]$ (so that, again, $\mathbb{E}[S] = \frac{1}{\mu}$).

Problem 3. Consider again the flip-flop process $X = \{X_t : t \geq 0\}$ from Problem 1 of Homework 6. Namely, assume that $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity λ, and define the flip-flop process $X = \{X_t : t \geq 0\}$ with state space $S = \{0, 1\}$ by

$$X_t = \frac{1}{2} + (-1)^{N_t}\left(X_0 - \frac{1}{2}\right) ,$$

where X_0 is a random variable with values in S that is independent of the process N. This complicated formula simply means that the process X switches between 0 and 1 at each event of N. Since N is a time-homogeneous Markov chain, X is also a time-homogeneous Markov chain. Assume that the initial state X_0 of the Markov chain X is a random variable with distribution

$$p_j(0) = \mathbb{P}(X_0 = j) = \frac{1}{2} \qquad \text{for } j = 0, 1 .$$

Note that $\mathbf{p}(0) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the stationary distribution of the chain X, as you found in Homework 6. Recall that in Homework 6 you have found (in several ways) the stochastic semigroup $\mathbf{P}_t$ of the continuous-time Markov chain X:

$$\mathbf{P}_t = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+\mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1-\mathrm{e}^{-2\lambda t}) \\ \frac{1}{2}(1-\mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1+\mathrm{e}^{-2\lambda t}) \end{pmatrix} .$$

(a) Show the distribution of the random variable X_t is $\mathbf{p}(t) = \begin{pmatrix} p_0(t) & p_1(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (where $p_i(t) = \mathbb{P}(X_t = i)$). You should do this in two ways: (1) by a calculation using the explicit expressions for $\mathbf{P}_t$ and $\mathbf{p}(0)$, and (2) by giving a simple reason why this is the answer, without any calculations.

(b) For $s \geq 0$, $t \geq 0$, show that the expectation of X_t is $m_X(t) := \mathbb{E}[X_t] = \frac{1}{2}$, and the autocorrelation function is $R_X(s, s+t) := \mathbb{E}[X_s X_{s+t}] = \frac{1}{4}\left(1 + \mathrm{e}^{-2\lambda t}\right)$.

Hint: To compute the expected value of the product $X_s X_{s+t}$, use the Tower Rule by conditioning on the state of the system at the earlier time, namely, $\mathbb{E}[X_s X_{s+t}] = \mathbb{E}[\mathbb{E}[X_s X_{s+t} | X_s]]$. You can first compute

$$\mathbb{E}[X_s X_{s+t} | X_s = j] = j\,\mathbb{E}[X_{s+t} | X_s = j] = j \sum_{k=0}^{1} k\, p_{jk}(t) = \cdots ,$$

and then apply the Tower Rule,

$$\mathbb{E}[X_s X_{s+t}] = \mathbb{E}[\mathbb{E}[X_s X_{s+t} | X_s]] = \sum_{k=0}^{1} \mathbb{E}[X_s X_{s+t} | X_s = j]\, \mathbb{P}(X_s = j) = \cdots .$$

(c) Is the process X weakly stationary? Explain briefly.

(d) Compute the autocovariance function

$$C_X(s, s{+}t) := \mathrm{Cov}\,(X_s, X_{s+t}) = \mathbb{E}[(X_s - \mathbb{E}X_s)(X_{s+t} - \mathbb{E}X_{s+t})] = R_X(s, s{+}t) - m_X(s)\, m_X(s{+}t) ,$$

and show that the autocorrelation function is equal to $\mathrm{e}^{-2\lambda t}$.

(e) Define the random process

$$Y_t := \int_0^t X_u \,\mathrm{d}u , \qquad t \geq 0 .$$

Show that $\mathbb{E}[Y_t] = \frac{t}{2}$ and $\mathbb{E}[Y_t^2] = \frac{t^2}{4} + \frac{t}{4\lambda} - \frac{1}{8\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right)$.

Hint: The expectation of the integral over t is equal to the integral over t of the expectation of the integrand, so that

$$\mathbb{E}[Y_t] = \mathbb{E}\left[\int_0^t X_u \,\mathrm{d}u\right] = \int_0^t \mathbb{E}\left[X_u\right] \mathrm{d}u = \cdots .$$

For $\mathbb{E}[Y_t^2]$, you can write

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \mathbb{E}\left[\left(\int_0^t X_u \,\mathrm{d}u\right)\left(\int_0^t X_v \,\mathrm{d}v\right)\right] = \mathbb{E}\left[\int_0^t \int_0^t X_u X_v \,\mathrm{d}u \,\mathrm{d}v\right] \\
&= \int_0^t \left(\int_0^u \mathbb{E}\left[X_u X_v\right] \,\mathrm{d}v\right) \mathrm{d}u + \int_0^t \left(\int_0^v \mathbb{E}\left[X_u X_v\right] \,\mathrm{d}u\right) \mathrm{d}v \\
&\qquad \text{[where we split the square } [0,t]\times[0,t] \text{ in two parts]} \\
&= 2\int_0^t \left(\int_0^v \mathbb{E}\left[X_u X_v\right] \,\mathrm{d}u\right) \mathrm{d}v \ ;
\end{aligned}$$

now notice that in the last integral the integration is over $0 \le u \le v \le t$, and recall that in part (b) you found that $\mathbb{E}\left[X_u X_v\right] = \frac{1}{4}\left(1 + \mathrm{e}^{-2\lambda(v-u)}\right)$ for $v > u$.

(f) Find $\operatorname{var} Y_t$ for the random process defined in part (e). Does $\operatorname{var} Y_t$ behave reasonably in the limit $t \to 0^+$? Explain.

Food for Thought Problem 1. Use the definition of the Dirac δ_a-function and its derivatives, namely,

$$\int_{\mathbb{R}} \delta_a(x)\, f(x)\,\mathrm{d}x := \delta_a(f) := f(a) \ , \qquad (\delta_a^{(k)})(f) := (-1)^k\, f^{(k)}(a) \ ,$$

to show that

$$\int_{\mathbb{R}} x^3\, \delta_2(x)\,\mathrm{d}x = 8 \ , \quad \int_{\mathbb{R}} \mathrm{e}^{5x^2}\, \delta_2(x)\,\mathrm{d}x = \mathrm{e}^{20} \ , \quad \int_{\mathbb{R}} \mathrm{e}^{5x^2}\, \delta_2'(x)\,\mathrm{d}x = 10\,\mathrm{e}^{20} \ .$$

Food for Thought Problem 2. Let $a > 0$ be an arbitrary number, and

$$u_a(x) = u(x-a) = \begin{cases} 0 \ , & x < a, \\ 1 \ , & x > a \ , \end{cases}$$

be the Heaviside function with a jump at a. By definition, the kth derivative of a generalized function ξ is defined by

$$(\xi^{(k)})(f) := (-1)^k \xi(f^{(k)}) \ ,$$

where f is an arbitrary test function. One can consider u_a as a generalized function defined by

$$u_a(f) := \int_{\mathbb{R}} u_a(x)\, f(x)\,\mathrm{d}x \ .$$

Show that $u_a' = \delta_a$, i.e., that for an arbitrary test function f

$$(u_a')(f) := \delta_a(f) = f(a) \ .$$

You can proceed as follows:

$$(u_a')(f) = -u_a(f') = -\int_{\mathbb{R}} u_a(x)\, f'(x)\,\mathrm{d}x = -\int_a^\infty f'(x)\,\mathrm{d}x = -\lim_{x\to\infty} f(x) + f(a) = \cdots \ .$$