MATH 5453 Homework 8 Due Thu, Oct 30

Problems 19, 26, 28, 31(c,d) from Section 2.3 of the book.

Hint to Problem 2.3/26: Use the Dominated Convergence Theorem.

General Hint to Problem 2.3/28: In a problem like this, use the fact that, for a bounded function on a compact interval, Riemann integrability implies Lebesgue measurability (and, therefore, Lebesgue integrability) of the function, and the Lebesgue integral is equal to the Riemann integral; if the domain of integration is infinite, you may need to think of the integral as a limit of integrals over increasing finite domains (see p. 57 of the book). Therefore, one strategy for finding $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$ is to find a function $g \in L^1$ such that $|f_n| \le g$ over the domain of integration, and then apply the Dominated Convergence Theorem. (Sometimes the Monotone Convergence Theorem will also work.) *Please, specify which theorem you use at each step of your solution!*

Hint to Problem 2.3/28(a): Use the binomial formula to expand $(1+\frac{x}{n})^n$ and obtain an estimate like

$$|f_n(x)| \le \frac{1}{(1+\frac{x}{n})^n} \le \frac{1}{1+\frac{x^2}{4}} \ ,$$

where the last inequality holds for any $n \ge 2$. Use this to justify the application of the Dominated Convergence Theorem. (Would the Monotone Convergence Theorem work here?)

Hint to Problem 2.3/28(b): Perhaps the so-called Bernoulli inequality, $1+n\xi \le (1+n\xi)^n$ (for $\xi \ge 0$, $n \in \mathbb{N}$), will be useful.

Hint to Problem 2.3/28(c): Recall that $|\sin\xi| \le |\xi|$.

Hint to Problem 2.3/28(d): An appropriate change of variables in the integral makes the problem very easy.

Hint to Problem 2.3/31(c): The following trick may help: write $\frac{1}{\mathrm{e}^x-1}$ as $\mathrm{e}^{-x}\frac{1}{1-\mathrm{e}^{-x}}$, and then use the formula for the sum of a geometric series, $1+z+z^2+z^3+\cdots = \frac{1}{1-z}$ (for $|z|<1$), to expand $\frac{1}{1-\mathrm{e}^{-x}}$. Use the definition of the gamma function on page 58.

Hint to Problem 2.3/31(d): Expand $\sin x$ in a Taylor series, and then integrate term-by-term. You may need to use the fact that $\Gamma(n+1)=n!$ for $n=0,1,2,\cdots$, and that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \ ,$$

which can be easily obtained by a term-by-term integration in

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \int_0^x \frac{1}{1-(-t^2)}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty}(-t^2)^n\,\mathrm{d}t \ .$$

Please turn the page!

Additional problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and the measure μ be finite (i.e., $\mu(X) < \infty$). Let $\mathscr{F}$ be the space of all measurable real-valued functions on X that are finite μ-a.e., with the addition and multiplication by a scalar defined in the usual way:

$$(f+g)(x) := f(x) + g(x) \ , \qquad (\alpha f)(x) := \alpha\, f(x) \ , \quad \alpha \in \mathbb{R} \ .$$

(a) Let a, b, and c be a non-negative numbers, satisfying $c \leq a + b$. Show by a direct calculation that

$$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b} \ .$$

(b) Use your result from (a) to show that the function $\delta : \mathscr{F} \times \mathscr{F} \to [0, \infty)$ by

$$\delta(f,g) = \int \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, \mathrm{d}\mu(x) \ , \qquad f, g \in \mathscr{F} \ ,$$

satisfies the triangle inequality, $\delta(f,h) \leq \delta(f,g) + \delta(g,h)$, for any $f, g, h \in \mathscr{F}$.

(c) Explain why $\delta(f,g) = 0$ if and only if $f = g$ μ-a.e. (Which theorem from the book guarantees this?) Let us define the equivalence relation $f \sim g$ iff $f = g$ μ-a.e. (Think how you would show that this is indeed an equivalent relation, but there is no need to write it in your homework.) Let $\mathsf{F} := \mathscr{F} / \sim$, and let $[f] \in \mathsf{F}$ be the equivalence class of the function $f \in \mathscr{F}$. Define the function $\Delta : \mathsf{F} \times \mathsf{F} \to [0, \infty)$ by

$$\Delta([f],[g]) := \delta(f,g) \ .$$

Explain why this function is well-defined (i.e., independent of the arbitrariness in the choice of a representative from an equivalence class), and show that it is a metric on the linear space F.

(d) Prove that a sequence $\{[f_n]\}_{n=1}^{\infty}$ in F converges to $[f] \in \mathsf{F}$ if and only if $f_n \to f$ in measure.

Hint: For any $\epsilon > 0$ and $n \in \mathbb{N}$ define the set $E_n(\epsilon) := \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$. Show that

$$\frac{\epsilon}{1+\epsilon} \mu(E_n(\epsilon)) \ \leq \ \delta(f_n, f) \ \leq \ \mu(E_n(\epsilon)) + \epsilon \mu(X) \ .$$

To prove these inequalities, you may need to use that $\int_X = \int_{E_n(\epsilon)} + \int_{E_n(\epsilon)^c}$, that $0 \leq \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} \leq 1$, that $\int_X \phi \, \mathrm{d}\mu \geq \int_{E_n(\epsilon)} \phi \, \mathrm{d}\mu$ for any nonnegative function $\phi \in L^1(X, \mu)$, and that the function $t \mapsto \frac{t}{1+t}$ is increasing for $t > -1$. To finish the proof, think how the behavior of $E_n(\epsilon)$ is related to the convergence of f_n to f in measure.

(e) **Food for thought:** Use the result of (d) to show that F is a complete metric space.

Hint: How is this related with Theorem 2.30 from the book?