MATH 4163 **Homework 8** **Due Thu, 03/29/2018**

Problem 1. [Even and odd functions; addition formulae for hyperbolic functions]
As we learned in class, each function $f : \mathbb{R} \to \mathbb{R}$ can be written as a sum of an even function f_{e} and an odd function f_{o}, where

$$f_{\mathrm{e}}(x) = \frac{f(x) + f(-x)}{2} , \qquad f_{\mathrm{o}}(x) = \frac{f(x) - f(-x)}{2} .$$

(a) Prove that, if f_{e} is an even function and f_{o} is an odd function, then the products $f_{\mathrm{e}} \cdot f_{\mathrm{e}}$ and $f_{\mathrm{o}} \cdot f_{\mathrm{o}}$ are even functions, while $f_{\mathrm{e}} \cdot f_{\mathrm{o}}$ is an odd function.

(b) Let $g(x) = 3 + 7x^2 - 5x^7 + \sin^3 x + \cos 7x - \mathrm{e}^{2x}$. Write down $g_{\mathrm{e}}(x)$ and $g_{\mathrm{o}}(x)$. No explanation is needed. (Recall that cos is even while sin is odd.)

(c) If $f(x) = \mathrm{e}^x$, then f_{e} and f_{o} are called *hyperbolic cosine* and *hyperbolic sine*, resp.:

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} , \qquad \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} .$$

In the identity $\mathrm{e}^{x+y} = \mathrm{e}^x \mathrm{e}^y$, use that $\mathrm{e}^x = \cosh x + \sinh x$, $\mathrm{e}^y = \cosh y + \sinh y$, and $\mathrm{e}^{x+y} = \cosh(x+y) + \sinh(x+y)$, apply your result from part (b) to identify the even and the odd parts in the two sides of the resulting identity, and obtain the addition formulae for hyperbolic sine and cosine

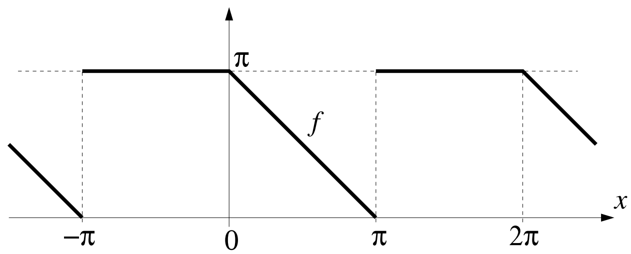
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y ,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y .$$

Problem 2. [Convergence of Fourier series; interesting identities]
Let f be a periodic function of period 2π which for $x \in (-\pi, \pi]$ is defined as

$$f(x) = \begin{cases} \pi , & -\pi < x \le 0 , \\ \pi - x , & 0 < x \le \pi ; \end{cases}$$

the graph of f is sketched in the figure below.
The Fourier series of f is the following (you do <u>not</u> have to prove this!):

$$\begin{aligned} f(x) \;\; = \;\; & \frac{3\pi}{4} + \frac{2}{\pi} \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots \right) \\ & - \sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots . \end{aligned} \tag{1}$$



(a) Use (1) to prove the identity $\displaystyle\sum_{\text{odd positive } n} \frac{1}{n^2} = \frac{\pi^2}{8}.$

(b) What identity do you obtain if you set $x = \dfrac{\pi}{2}$ in (1)?

(c) If you plug $x = \pi$ in the Fourier series in the right-hand side of (1), will the Fourier series converge (just say "yes" or "no")? If it converges, to what value does it converge? (Do not attempt to compute the value of the infinite sum, just write a couple of sentences about what the general theory predicts.)

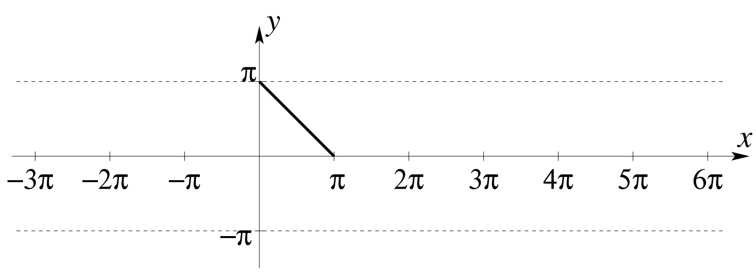
Problem 3. [Fourier sine and cosine series]

(a) Extend the function

$$f(x) = \pi - x \quad \text{for } x \in (0, \pi) \tag{2}$$

(shown in the figure below) as an *even* periodic function $f_{\text{even}}(x)$ of period 2π defined on $\mathbb{R}$. Explain briefly how you did it. Draw the graph of $f_{\text{even}}(x)$ for $x \in \mathbb{R}$.

Hint: Although f is not defined at 0 and π, you can choose values of $f_{\text{even}}(x)$ at $x = n\pi$ $(n \in \mathbb{Z})$ so that $f_{\text{even}}(x)$ be continuous.



(b) *Without doing any computations,* from the expressions below choose the one that gives the Fourier series of the function $f_{\text{even}}(x)$ obtained in part (a).

$$f_{\text{even}}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \cos nx$$

$$f_{\text{even}}(x) = -\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \cos nx$$

$$f_{\text{even}}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \sin nx$$

$$f_{\text{even}}(x) = -\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \sin nx$$

$$f_{\text{even}}(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n^2} \cos nx - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx$$

Give the reasons for your choice – a couple of sentences only, but *please be concrete*!
Hint: One of the Fourier coefficients of a periodic function is related to the average value of the function.

(c) Extend the function $f : (0, 1) \to \mathbb{R}$ defined in (2) as an *odd* periodic function $f_{\text{odd}}(x)$ of period 2π defined on $\mathbb{R}$. What does the fact that $f_{\text{odd}}(x)$ is an odd function imply about the value of $f_{\text{odd}}(0)$? Draw the graph of $f_{\text{odd}}(x)$.

(d) The Fourier series of the function $f_{\text{odd}}(x)$ obtained in part (c) is

$$f_{\text{odd}}(x) = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \ .$$

Suppose that you want to extend the function $f(x) = \pi - x$ for $x \in (0, \pi)$ to the whole real line and then to find the Fourier series of the extension, with the purpose of computing the values of $f(x)$ for $x \in (0, \pi)$ numerically by approximating this value by a partial sum of the corresponding Fourier series. Which expansion will be more beneficial numerically – the one of $f_{\text{even}}(x)$ from part (b) or the one of $f_{\text{odd}}(x)$ given above – if you want to use as few as possible terms in the series in order to achieve give accuracy of your answer? Explain your choice with one sentence only.

(e) Looking at the graphs of $f_{\text{even}}(x)$ from part (b) and $f_{\text{odd}}(x)$ from part (c), can you explain the decay of the Fourier coefficients in the Fourier series for $f_{\text{even}}(x)$ and $f_{\text{odd}}(x)$ from the properties of these functions (discontinuous, continuous, differentiable, etc.)?

Problem 4. [Using Fourier series in separation of variables]

In all parts of the problem below, you can use *without deriving* the following solutions of the heat equation $u_t(x, t) = \alpha^2 u_{xx}(x, t)$, $x \in [0, L]$, $t \geq 0$, with appropriate boundary conditions;

the first expression is for zero temperature at both boundaries (homogeneous Dirichlet BCs, $u(0,t) = 0$, $u(L,t) = 0$), and the second is for zero heat flux at both boundaries (homogeneous Neumann BCs, $u_x(0,t) = 0$, $u_x(L,t) = 0$):

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \exp\left\{-\left(\frac{\alpha n\pi}{L}\right)^2 t\right\} \sin\frac{n\pi x}{L} \ , \\ u(x,t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left\{-\left(\frac{\alpha n\pi}{L}\right)^2 t\right\} \cos\frac{n\pi x}{L} \ . \end{aligned}$$

(a) Solve the Dirichlet IBVP below to find the temperature $u(x,t)$.

$$\begin{aligned} &u_t = 3^2 u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 4\sin 2x + 7\sin 5x \ . \end{aligned}$$

In the expression for $u(x,t)$ take the limit $t \to \infty$ to find the asymptotic temperature, $u_\infty(x) := \lim_{t\to\infty} u(x,t)$. Explain why the expression you obtained for $u_\infty(x)$ is physically obvious.

(b) Use the formulae for products of trigonometric functions given in Problem 2(b) of Homework 4, to solve the following Dirichlet IBVP:

$$\begin{aligned} &u_t = 3^2 u_{xx} \ , \qquad x \in [0,\pi] \ , \quad t \geq 0 \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 4\sin 4x \cos 2x \ . \end{aligned}$$

(c) Solve the Neumann IBVP below.

$$\begin{aligned} &u_t = 3^2 u_{xx} \ , \qquad x \in [0,2] \ , \quad t \geq 0 \ , \\ &u_x(0,t) = 0 \ , \quad u_x(2,t) = 0 \ , \\ &u(x,0) = -1 + 7\cos\frac{5\pi x}{2} \ . \end{aligned}$$

(d) Solve the Neumann IBVP below.

$$\begin{aligned} &u_t = 3^2 u_{xx} \ , \qquad x \in [0,2] \ , \quad t \geq 0 \ , \\ &u_x(0,t) = 0 \ , \quad u_x(2,t) = 0 \ , \\ &u(x,0) = f(x) := \begin{cases} x & \text{for } x \in [0,1] \ , \\ 2 - x & \text{for } x \in [1,2] \ . \end{cases} \end{aligned}$$

You may use either the sine or the cosine Fourier expansion (you choose which one) of the function f from the initial condition, given below (you do *not* need to derive them):

$$f(x) = \frac{8}{\pi^2}\left(\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \frac{1}{7^2}\sin\frac{7\pi x}{2} + \cdots\right)$$
$$= 1 - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{6\pi x}{2} + \frac{1}{5^2}\cos\frac{10\pi x}{2} + \frac{1}{7^2}\cos\frac{14\pi x}{2} + \cdots\right) .$$

Problem 5. [Wave equation with strong air resistance]

As we learned in class, the motion of a string in a vertical plane is described by the PDE

$$\rho(x)\, u_{tt} = \frac{\partial}{\partial x}\left[\tau(x)\, u_x\right] - \gamma\, u_t - \rho(x) g \ . \tag{3}$$

The meaning of the symbols is the following:

- the function $u(x,t)$ (measured in meters) describes the vertical coordinate z of the point of the string with spatial coordinate x at time t, i.e., $z = u(x,t)$;
- $\rho(x)$ is the linear density of the string, i.e., the mass per unit length (unit for ρ: $\frac{\mathrm{kg}}{\mathrm{m}}$);
- $\tau(x)$ is the tension in the string (unit: Newton $= \frac{\mathrm{kg\cdot m}}{\mathrm{s}^2}$);
- $\gamma > 0$ is a quantity describing the resistance force acting on a unit length of the string (unit: $\frac{\mathrm{kg}}{\mathrm{m\cdot s}}$);
- g is the free-fall acceleration (unit: $\frac{\mathrm{m}}{\mathrm{s}^2}$).

All the quantities $\rho(x)$, $\tau(x)$, γ, and g are positive.

If ρ and τ are constants, (3) is often written in the form

$$u_{tt} = c^2\, u_{xx} - \beta\, u_t - g \ ,$$

where $c = \sqrt{\frac{\tau}{\rho}}$ is a positive constant equal to the speed of propagation of the disturbances along the string, and $\beta = \frac{\gamma}{\rho}$ is a positive constant.

In this problem you will solve the following initial boundary value problem for the wave equation:

$$\begin{aligned}
&u_{tt} = \frac{1}{4}\, u_{xx} - 5u_t \ , \qquad (x,t) \in [0,\pi] \times \mathbb{R}_+ \ , \\
&u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\
&u(x,0) = 4\, \sin 4x \ , \\
&u_t(x,0) = -7\, \sin 4x \ ,
\end{aligned} \tag{4}$$

which describes waves in a string with fixed ends in presence of a resistance force.

(a) Separate variables, i.e., assume that the function u has the form

$$u(x,t) = X(x)\,T(t) \ .$$

Plug this form of u in the PDE in (4) and derive the ODEs that the functions $X(x)$ and $T(t)$ must satisfy.

(b) As you must have obtained in part (a), the function $X(x)$ must satisfy the BVP

$$\begin{aligned} &X''(x) - \mu X(x) = 0 \ , \qquad x \in [0,\pi] \ , \\ &X(0) = 0 \ , \quad X(\pi) = 0 \ , \end{aligned}$$

Write down the functions $X_n(x)$ and the values μ_n that solve this problem; there is no need to derive them again.

(c) Show that the functions $T_n(t)$ satisfy the ODEs

$$T_n''(t) + 5\,T_n'(t) + \frac{n^2}{4}\,T_n(t) = 0 \ , \qquad n = 1,2,3,\dots \ . \tag{5}$$

(d) Write the expansion of $u(x,t)$ in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin nx \ , \tag{6}$$

and derive the initial conditions for the functions $T_n(t)$; note that you have to find the values of both $T_n(0)$ and $T_n'(0)$.

(e) Since all functions $T_n(t)$ satisfy the homogeneous linear ODEs (5) ("homogeneous" in the sense that the right-hand sides of the ODEs are 0), and for all values of n except $n = 4$ the initial conditions are zero, all functions $T_n(t)$ except $T_4(t)$ are identically zero. Write down and solve the initial value problem for the function $T_4(t)$.

(f) Write down the solution $u(x,t)$ of the IBVP (4).

Remark: In the expression for $u(x,t)$ that you have obtained, note that the function $T_4(t)$ does not oscillate, so that this contribution to the movement of the string does not correspond to a wave. This happens because the dissipation of energy is too strong.

Problem 6. [Wave equation with weak air resistance]

In this problem you will solve an IBVP very similar to the one given in (4), but with different initial conditions, namely,

$$\begin{aligned} &u_{tt} = \frac{1}{4}\,u_{xx} - 5u_t \ , \qquad (x,t) \in [0,\pi]\times\mathbb{R}_+ \ , \\ &u(0,t) = 0 \ , \quad u(\pi,t) = 0 \ , \\ &u(x,0) = 2\,\sin 13x \ , \\ &u_t(x,0) = \sin 13x \ , \end{aligned} \tag{7}$$

While the method for solving (7) is the same as in Problem 5, the physical meaning of the solution will be different.

(a) All the results obtained in parts (a)–(c) of Problem 5 would hold for the IBVP (7), so you do not need to rederive them here. As in Problem 5(d), the solution of (7) can be written as an expansion of the form (6). Plug (6) into the initial conditions in (7) to obtain the initial conditions for the functions $T_n(0)$.

(b) As in Problem 5(e), only for one value of n, namely, for $n = 13$, the value of $T_n(0)$ will be non-zero, so that all functions but $T_n(t)$ with $n \neq 13$ will be identically zero. Find the general solution of the ODE (5) for $T_{13}(t)$.

Hint: Recall that, if the roots of the characteristic equations are $\alpha \pm \mathrm{i}\beta$, then the general solution of the ODE is $\mathrm{e}^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$.

(c) Impose the initial conditions found in part (a) on the general solution for $T_{13}(t)$ found in part (b), to find the function $T_{13}(t)$.

(d) Write down the solution $u(x,t)$ of the IBVP (7).

Remark: In the expression for $u(x,t)$ that you have obtained, the functions $T_{13}(t)$ oscillate with exponentially decreasing amplitude, so that the solution $u(x,t)$ corresponds to a wave that loses energy while propagating, due to the energy dissipated by the air resistance.