# MATH 4073 Homework 8 Due $\boxed{\text{in class}}$ on Mon, 11/14/16

**Problem 1.** The Trapezoidal Rule applied to $\int_0^2 f(x)\,dx$ gives the value 4, and Simpson's Rule gives the value 2. What is $f(1)$?

**Problem 2.** The quadrature formula

$$\int_0^2 f(x)\,\mathrm{d}x = c_0 f(0) + c_1 f(1) + c_2 f(2)$$

is exact for all polynomials of degree less than or equal to 2. Determine $c_0$, $c_1$, and $c_2$.

*Hint:* Note that this quadrature formula must be exact for the polynomials $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$. Use this fact to write a system of three (linear) equations for $c_0$, $c_1$, and $c_2$.

**Problem 3.** Mr. Tran defined a family of polynomials which he (very modestly) denoted by $T_0$, $T_1$, $T_2$, .... These polynomials satisfy the following conditions:

(i) the polynomial $T_k$ is of degree $k$;

(ii) $T_0(x) = 1$, and for all $k = 1, 2, 3, \ldots$, the coefficient of $x^k$ in $T_k$ is equal to $2^{k-1}$;

(iii) the polynomials $T_0$, $T_1$, $T_2$, ..., $T_n$ form an orthogonal basis in the space of polynomials $V_n\left(-1, 1;\ w(x) = \frac{1}{\sqrt{1-x^2}}\right)$.

Recall that $V_n(a, b; w(x))$ stands for the linear space of polynomials of degree $\le n$ endowed with the inner product

$$\langle P, Q\rangle = \int_a^b P(x)\,Q(x)\,w(x)\,\mathrm{d}x\ .$$

In the solution of this problem the following identities will be handy:

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx &= \pi\ , \\
\int_{-1}^{1} \frac{x^{2m}}{\sqrt{1-x^2}}\,dx &= \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2m-1)}{2^m m!}\,\pi \qquad \text{for } m = 1, 2, 3, \ldots\ , \\
\int_{-1}^{1} \frac{x^{2m-1}}{\sqrt{1-x^2}}\,dx &= 0\ , \qquad \text{for } m = 1, 2, 3, \ldots \text{ (obviously!)}\ .
\end{aligned}$$

*Please show your calculations in detail!*

(a) Find the only polynomial $T_1$ of degree 1 of the form $T_1(x) = x + \cdots$ that is orthogonal to $T_0$ (recall that $T_0(x) = 1$ by definition).

(b) Find the only quadratic polynomial $T_2$ of the form $T_2(x) = 2x^2 + \cdots$ that is orthogonal to both $T_0$ and $T_1$.

(c) Show that the polynomial $P(x) = 6x^2 - 5x + 4$ can be represented as a linear combination of the polynomials $T_0$, $T_1$ and $T_2$ as follows: $P = 3T_2 - 5T_1 + 7T_0$.

(d) Find the orthogonal projection, $\mathrm{proj}_{T_0+2T_1} P$, of the polynomial $P(x) = 6x^2 - 5x + 4$ onto the " straight line"

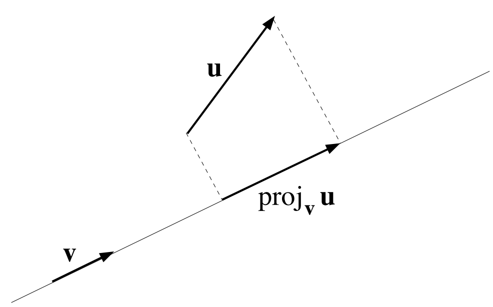
$$\ell := \{t(T_0 + 2T_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2\left(-1, 1; \frac{1}{\sqrt{1-x^2}}\right)$. If you have solved the previous parts of this problem, finding this orthogonal projection should be easy.

*Hint:* If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space $V$, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\,\mathbf{v}$$

as in the picture below.




(e) Find the general form of all polynomials that are orthogonal to the polynomial $P = 3T_2 - 5T_1 + 7T_0$ in the sense of the inner product used in this problem. In more geometric terminology, find the subspace of the inner product linear space $V_2\left(-1, 1; \frac{1}{\sqrt{1-x^2}}\right)$ that is orthogonal to $P$.

**Problem 4.** This problem is an application of Mr. Tran's polynomials $T_k$ studied in Problem 3 to Gaussian quadrature. The integrals given there will be useful for this problem as well. In this problem you will use the fact that the polynomials

$$T_0(x) = 1\ , \quad T_1(x) = x\ , \quad T_2(x) = 2x^2 - 1\ , \quad T_3(x) = 4x^3 - 3x$$

form an orthogonal basis in the inner product linear space $V_3\left(-1, 1; w(x) = \frac{1}{\sqrt{1-x^2}}\right)$ of all polynomials of degree $\leq 3$ endowed with the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$.

The goal of this problem is to find a Gaussian quadrature formula with degree of precision 5 based on the general formalism developed in class. The notations used are the same as in the handout *Theoretical foundations of Gaussian quadrature*. Because of the specific form of the weight function, the formula you will develop will be particularly suitable for computing integrals of the form $\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$.

In all parts of this problem, you need only to compute the quantities that are not given (e.g., in part (a) you only need to find $x_3$, in part (b) only $L_3(x)$, in part (c) only $w_3$). *Please show your calculations in detail!*

(a) Find the roots $x_1 < x_2 < x_3$ of the polynomial $T_3$.

*Hint:* I found that $x_1 = -\frac{\sqrt{3}}{2}$ and $x_2 = 0$.

(b) Write down the polynomials $L_1$, $L_2$, $L_3$.

*Hint:* I obtained the following for $L_1$ and $L_2$: $L_1(x) = \frac{2}{3}x^2 - \frac{1}{\sqrt{3}}x$, $L_2(x) = 1 - \frac{4}{3}x^2$.

(c) Find the weights $w_1$, $w_2$, $w_3$.

*Hint:* I computed that $w_1 = \frac{\pi}{3}$, $w_2 = \frac{\pi}{3}$.

(d) Write down the quadrature formula coming from parts (a), (b), (c).

(e) Show that the quadrature formula obtained in (d) is *exact* for all polynomials of degree 5.

(f) Show that the quadrature formula obtained in (d) is *not* exact for the polynomial $f(x) = x^6$. Does this agree with the theoretical prediction about the degree of precision of the method you developed?

(g) Now apply the beautiful quadrature formula you derived in (d) to compute the approximate value of the integral

$$\int_{-1}^{1} \frac{\mathrm{d}x}{(2-x)\sqrt{1-x^2}} \ ,$$

whose exact value is $\frac{\pi}{\sqrt{3}}$. Find the numerical values of the absolute and the relative error.