

Problem 1. In this problem you will apply several different methods to approximate the first derivative of the function $f(x) = \sin x$ at $x = \frac{\pi}{3}$.

- (a) Apply the forward 2-point formula

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

with $h = 0.01$ to find the approximate value of $f'(\frac{\pi}{3})$. Find the rigorous error bound,

$$\text{Error} \leq \frac{h}{2} \max_{\xi \in [x_0, x_0+h]} |f''(\xi)|$$

(see equation (4.1) in the book). Compute the true value of the derivative $f'(\frac{\pi}{3})$, find the true absolute error and compare with the rigorous bound.

- (b) Apply the 3-point formula

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

with $h = 0.01$ to find the approximate value of $f'(\frac{\pi}{3})$. Find the rigorous error bound given by equation (4.5) in the book. Find the true value of the absolute error and compare with the rigorous bound.

Problem 2. Let $N_1(h)$ be an approximation to M for every $h > 0$ and

$$M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \dots$$

for some constants K_1, K_2, K_3, \dots . Suppose that that you know the values of $N_1(h)$, $N_1(\frac{h}{3})$, and $N_1(\frac{h}{9})$.

- (a) Derive a formula for $N_2(h)$ which is based on the values of $N_1(h)$ and $N_1(\frac{h}{3})$ and provides an $\mathcal{O}(h^2)$ approximation to M .
- (b) Derive a formula for $N_3(h)$ which is based on the values of $N_2(h)$ and $N_2(\frac{h}{3})$ and provides an $\mathcal{O}(h^3)$ approximation to M .
- (c) Let $f(x)$ be a smooth function. Use the expansion of $f(x_0 + h)$ in a Taylor series around x_0 to derive the formula

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{f''(x_0)}{2} h - \frac{f'''(x_0)}{6} h^2 - \frac{f^{(4)}(x_0)}{24} h^3 - \dots$$

Now let M stand for the exact value of $f'(x_0)$ and $N(h) := \frac{f(x_0 + h) - f(x_0)}{h}$. Explain briefly why this implies that Richardson extrapolation can be applied to the forward-difference formula for approximating first derivatives (equation (4.1) in the book).

- (d) Apply the formulae obtained in (a) and (b) to compute the $\mathcal{O}(h^2)$ and $\mathcal{O}(h^3)$ approximations to the derivative of the function $f(x) = \sin x$ at $x_0 = \frac{\pi}{3}$ for $h = 0.09$ by using Richardson's extrapolation applied to the forward-difference formula for $f'(x_0)$. Compute the numerical values of the actual errors, $|N_j(0.09) - f'(\frac{\pi}{3})|$, for $j = 1, 2, 3$.

Problem 3. In this problem we will try to integrate a function approximately, without actually computing complicated integrals. Let

$$A(a, b) := \int_0^b \frac{x^4}{1 + a^2x^4} dx ,$$

where $a \in [0, 1]$ and $b \in [0, 0.6]$ are constants.

- (a) Prove that

$$\int_0^b \frac{x^4}{1 + a^2x^4} dx = \frac{b}{a^2} - \frac{1}{a^{5/2}} \left(\frac{1}{4\sqrt{2}} \ln \frac{1 + b\sqrt{2a} + ab^2}{1 - b\sqrt{2a} + ab^2} + \frac{1}{2\sqrt{2}} \arctan \frac{b\sqrt{2a}}{1 - ab^2} \right) .$$

No, I am just kidding – don't do it!!! But seriously, if your life depended on it, how would you approach this problem? If direct integration looks impossible, can you think of some other – cleverer – method of proving this formula? Just tell me what you are *planning* to do, do not perform the actual calculation!

- (b) Use that, for $x \in [0, b]$, $\frac{x^4}{1+a^2b^4} \leq \frac{x^4}{1+a^2x^4} \leq x^4$ to derive the bounds

$$\frac{b^5}{5(1 + a^2b^4)} \leq \int_0^b \frac{x^4}{1 + a^2x^4} dx \leq \frac{b^5}{5} ,$$

and apply these bounds to find numerical upper and lower bounds on the value of $A(1, 0.6)$.

- (c) Expand the integrand, $\frac{x^4}{1+a^2x^4}$, in a Taylor series around $x = 0$.

Hint: This is very easy: in part (c) you found that $|a^2x^4| < 1$, so you can use the formula for the Taylor series of $\frac{1}{1-q}$ around $q = 0$ (this formula is valid for $|q| < 1$), and then apply it to expand $\frac{1}{1+a^2x^4}$. You have probably learned the Taylor expansion of $\frac{1}{1-q}$ long before you heard the words "Taylor expansion".

- (d) Take the first three terms in the Taylor expansion of $\frac{x^4}{1+a^2x^4}$ and integrate them term by term to find the approximate numerical value of the integral. Find the absolute error of your computation, if you know that the exact value is

$$A(1, 0.6)_{\text{exact}} = 0.01452370832222581712414099495156847987 \dots .$$

Problem 4. This problem is about different methods for approximate integration applied to the integral $\int_0^{\pi/3} \tan x \, dx$. In this problem “trapezoidal rule” and “midpoint rule” always refer to the “simple” rules, i.e., the ones derived in Section 4.3 of the book (not the “composite” ones discussed in Section 4.4).

- (a) Compute by hand the indefinite integral $\int \tan x \, dx$; show me your solution in detail. Find the exact value of the definite integral $\int_0^{\pi/3} \tan x \, dx$.
- (b) Is the function $\tan x$ increasing on $[0, \frac{\pi}{3}]$? Is it concave up or concave down on $[0, \frac{\pi}{3}]$? Justify your answers.
- (c) Based on your answer in part (b), can you predict whether the trapezoidal rule applied to $\int_0^{\pi/3} \tan x \, dx$ will give you a smaller or a larger value than the true value of the integral (obtained in part (a))? Explain and draw a sketch to support your claim.
- (d) Apply the trapezoidal rule to the integral $\int_0^{\pi/3} \tan x \, dx$. Does your result agree with your prediction in part (c)?
- (e) Find the rigorous upper bound on the error of the trapezoidal rule in the computation of the integral $\int_0^{\pi/3} \tan x \, dx$.
- (f) Find the true value of the absolute error in the computation of $\int_0^{\pi/3} \tan x \, dx$ by using the trapezoidal rule, and compare it with the rigorous upper bound found in part (e).