MATH 4073 Homework 8 Due 11/12/09

Problem 1. Consider the function $f(x) = \sin \frac{\pi x}{6}$. In this you will construct its cubic spline interpolant $S(x)$ based on the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 3$. You have to impose a free boundary condition at the left end $x_0 = 0$, and a clamped one, $S'(x_2) = f'(x_2)$, at the right end $x_2 = 3$. Take

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3 & \text{for } x \in [0,1] \ , \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 & \text{for } x \in [1,3] \ . \end{cases}$$

(a) Write down the system of 8 equations for the 8 unknown coefficients a_j, b_j, c_j, d_j $(j = 0, 1)$. First write the conditions in the abstract form (without using the explicit expressions for S_0 and S_1 in terms of the coefficients a_j, b_j, c_j and d_j), and then write the equations again as a system of equations for the coefficients. Solve the equations; you will find the values of a_0, a_1 and c_0 without any effort. Since the remaining equations are more complicated, you can use that $d_0 = -\frac{1}{40}$. (Here are some values that you have to obtain: $c_1 = -\frac{3}{40}$, $d_1 = -\frac{1}{80}$.)

(b) Use Theorem 3.13 (page 152) to give a rigorous upper bound on the interpolation error. Compare the numerical value of this bound with the true error which is about 0.0035.

(c) Find the numerical value of $\int_0^2 S(x)\,\mathrm{d}x$ and compare it with the exact value, $\int_0^2 f(x)\,\mathrm{d}x = \frac{3}{\pi}$; find the absolute and the relative error.

(d) If you know the rigorous upper bound from part (b), how can you use it to find a rigorous upper bound on the error in the integration you did in part (c)? Find this bound and compare it with the true value of the absolute error found in part (c).

(e) Find the numerical value of $S'(2)$ and compare it with the exact value, $f'(2) = \frac{\pi}{12}$; find the absolute and the relative error.

Problem 2. Use the values of $f(x) = \tan x$ at $x = 0.895$, $x = 0.900$, and $x = 0.905$ radians, to compute the approximate values of $f'(0.9)$, the numerical values of the rigorous error bounds, and the actual errors by using each of the following three methods:

(a) the forward-difference formula;

(b) the backward-difference formula;

(c) the 3-point formula that uses only the values of $f(x_0 + h)$ and $f(x_0 - h)$; you may use the following fact:

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} \tan x = \frac{2}{\cos^2 x} \left(1 + 3\tan^2 x\right) \ .$$

Problem 3. Let $N_1(h)$ be an approximation to M for every $h > 0$ and

$$M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

for some constants K_1, K_2, K_3, ... Suppose that that you know the values of $N_1(h)$, $N_1(\frac{h}{4})$, and $N_1(\frac{h}{16})$.

(a) Derive a formula for $N_2(h)$ which is based on the values of $N_1(h)$ and $N_1(\frac{h}{4})$ and provides an $\mathcal{O}(h^2)$ approximation to M.

(b) Derive a formula for $N_3(h)$ which is based on the values of $N_2(h)$ and $N_2(\frac{h}{4})$ and provides an $\mathcal{O}(h^3)$ approximation to M.

(c) Let $f(x)$ be a smooth function. Use the expansion of $f(x_0 + h)$ in a Taylor series around x_0 to derive the formula

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{f''(x_0)}{2}\,h - \frac{f'''(x_0)}{6}\,h^2 - \frac{f^{(4)}(x_0)}{24}\,h^3 - \cdots .$$

Now let M stand for the exact value of $f'(x_0)$ and $N(h) := \dfrac{f(x_0 + h) - f(x_0)}{h}$. Explain briefly why this implies that Richardson extrapolation can be applied to the forward-differentce formula for approximating first derivatives (equation (4.1) in the book).

(d) Apply the formulae obtained in (a) and (b) to compute the $\mathcal{O}(h^2)$ and $\mathcal{O}(h^3)$ approximations to the derivative of the function $f(x) = \cos x$ at $x_0 = \frac{\pi}{6}$ for $h = 0.08$ by using Richardson's extrapolation applied to the forward-difference formula for $f'(x_0)$. Compute the numerical values of the actual errors, $|N_j(0.08) - f'(\frac{\pi}{6})|$, for $j = 1, 2, 3$.