

Problem 1. Consider the following vectors in \mathbb{R}^3 : $\mathbf{u} := \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$, $\mathbf{v} := \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$, and $\mathbf{w} := \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$. Directly from the definition, find out whether they are linearly dependent. Please write all your calculations in detail.

Problem 2. Let V be a two-dimensional linear space, and $\mathbf{A} : V \rightarrow V$ be a linear operator acting on V . Let the vectors \mathbf{f}_1 and \mathbf{f}_2 form a basis in V , and the linear operator \mathbf{A} acts on these two vectors as follows: $\mathbf{A}\mathbf{f}_1 = 2\mathbf{f}_1$, $\mathbf{A}\mathbf{f}_2 = 4\mathbf{f}_1 + \mathbf{f}_2$.

- (a) Write down the matrix $\underline{\underline{A}} = (a_{ij})$ of the operator \mathbf{A} in the basis $\mathbf{f}_1, \mathbf{f}_2$; recall that the entries of $\underline{\underline{A}}$ are defined by

$$\mathbf{A}\mathbf{f}_j =: \sum_{i=1}^n a_{ij} \mathbf{f}_i .$$

- (b) If $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$, find $\mathbf{A}\mathbf{u}$ (expressed as a linear combination of the vectors \mathbf{f}_1 and \mathbf{f}_2).
- (c) Show that the vectors $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$ and $\mathbf{v} = 2\mathbf{f}_1 + \mathbf{f}_2$ are linearly independent, i.e., that if $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ (where $\alpha, \beta \in \mathbb{R}$), then $\alpha = 0$ and $\beta = 0$.

Problem 3. Suppose that the linear operator $\mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ transforms $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ into $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ into $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ into $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Find the matrix $\underline{\underline{A}}$ that corresponds to the operator \mathbf{A} .

Hint: You may use that, if $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the standard basis in \mathbb{R}^3 , then $\mathbf{u}_1 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{A}\mathbf{u}_1 = \mathbf{v}_1 = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$, which, by using the fact that \mathbf{A} is a linear operator, is the same as $\mathbf{A}\mathbf{e}_1 + \mathbf{A}\mathbf{e}_3 = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$; similarly, you will obtain two more equations: $\mathbf{A}\mathbf{e}_2 + \mathbf{A}\mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{A}\mathbf{e}_1 + \mathbf{A}\mathbf{e}_2 = (\text{something})$, from which you can express $\mathbf{A}\mathbf{e}_j$ ($j = 1, 2, 3$) in terms of the vectors \mathbf{e}_i ($i = 1, 2, 3$).

Problem 4. Consider the linear space of polynomials of degree no greater than 3. Let us choose $\mathbf{f}_0 = 1$, $\mathbf{f}_1 = x$, $\mathbf{f}_2 = x^2$, $\mathbf{f}_3 = x^3$ to be a basis in this vector space, so that each polynomial $P(x) = p_0 + p_1x + p_2x^2 + p_3x^3$ can be written as a vector \mathbf{p} in this space as

$$\mathbf{p} = \sum_{i=0}^3 p_i \mathbf{f}_i .$$

- (a) Let \mathbf{D} be the differentiation operator, i.e., $\mathbf{D}\mathbf{p}$ is the polynomial $\frac{d}{dx}P(x)$. Find the matrix $\underline{\underline{D}}$ of \mathbf{D} in the basis $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$.
- (b) Find the matrix of the operator \mathbf{D}^2 (where $\mathbf{D}^2 := \mathbf{D}\mathbf{D}$). You may use the fact that the matrix of a composition of the operators \mathbf{A} and \mathbf{B} is equal to the product of the matrices of \mathbf{A} and \mathbf{B} .
- (c) Find the matrices of \mathbf{D}^k for $k = 3, 4, \dots$. Do you get the zero operator \mathbf{O} for some value of k ? An operator \mathbf{A} for which it happens that $\mathbf{A}^k = \mathbf{O}$ for some k is called a *nilpotent* operator.

Problem 5. Consider the set of matrices

$$\mathcal{H} := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\} .$$

- (a) Is the set \mathcal{H} closed under matrix multiplication? (In other words, does the product of two matrices from \mathcal{H} always belong to \mathcal{H} ?) Justify your answer.
- (b) Generally speaking, matrix multiplication is not commutative. Prove that if $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are matrices from \mathcal{H} , then $\underline{\underline{A}}\underline{\underline{B}} = \underline{\underline{B}}\underline{\underline{A}}$.

Food for Thought Problem 1¹ [Change of basis in a linear space]

Let V be a linear space without any additional structure on it (i.e., there is no norm, inner product, or the concept of orthogonality). Consider the linear operator $\mathbf{A} : V \rightarrow V$ acting on V . Let $\mathbf{f}_1, \dots, \mathbf{f}_n$ be a basis in V . One can define the matrix elements of \mathbf{A} in this basis as follows:

$$\mathbf{A}\mathbf{f}_j =: \sum_{i=1}^n a_{ij} \mathbf{f}_i ,$$

so that the matrix of \mathbf{A} in this basis is $\underline{\underline{A}} = (a_{ij})$ (see equation (7) on page 457).

- (a) Let $\mathbf{u} = \sum_{j=1}^n u_j \mathbf{f}_j$. Find the components of $\mathbf{A}\mathbf{u}$ in the basis $\mathbf{f}_1, \dots, \mathbf{f}_n$; in other words, if $\mathbf{v} = \mathbf{A}\mathbf{u}$, find the numbers v_i such that $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{f}_i$.

Solution:

$$\begin{aligned} \mathbf{v} = \mathbf{A}\mathbf{u} &= \mathbf{A} \sum_j u_j \mathbf{f}_j = \sum_j u_j \mathbf{A}\mathbf{f}_j \\ &= \sum_j u_j \sum_i a_{ij} \mathbf{f}_i = \sum_i \left(\sum_j a_{ij} u_j \right) \mathbf{f}_i . \end{aligned}$$

¹You do not need to turn in the solution of this problem, but you are expected to understand it completely.

Comparing this with $\mathbf{v} = \sum_i v_i \mathbf{f}_i$, we see that

$$v_i = \sum_j a_{ij} u_j .$$

Recalling the definition of a multiplication of two matrices, we can interpret this equality as follows: if the vectors \mathbf{u} and \mathbf{v} are written as column vectors (i.e., as $n \times 1$ matrices, where $n = \dim V$), and $\underline{\underline{A}} = (a_{ij})$ is an $n \times n$ matrix, then the column vector $\mathbf{v} = \mathbf{A}\mathbf{u}$ is the product of the matrix $\underline{\underline{A}}$ and the column vector \mathbf{u} .

- (b) Let \mathbf{B} be another linear operator with matrix elements (b_{ij}) in the basis $\mathbf{f}_1, \dots, \mathbf{f}_n$. Find the matrix elements of the operator \mathbf{AB} in the same basis. Here \mathbf{AB} stands the composition of the operators in the sense that $(\mathbf{AB})\mathbf{u} := \mathbf{A}(\mathbf{B}\mathbf{u})$.

Solution:

$$\begin{aligned} (\mathbf{AB})\mathbf{u} &= \mathbf{A}(\mathbf{B}\mathbf{u}) = \mathbf{A} \left(\mathbf{B} \sum_j u_j \mathbf{f}_j \right) = \mathbf{A} \left(\sum_j u_j \mathbf{B}\mathbf{f}_j \right) = \mathbf{A} \left(\sum_j u_j \sum_i b_{ij} \mathbf{f}_i \right) \\ &= \mathbf{A} \left(\sum_{j,i} u_j b_{ij} \mathbf{f}_i \right) = \sum_{j,i} u_j b_{ij} \mathbf{A}\mathbf{f}_i = \sum_{j,i} u_j b_{ij} \sum_k a_{ki} \mathbf{f}_k = \sum_{j,i,k} u_j b_{ij} a_{ki} \mathbf{f}_k \\ &= \sum_k \left[\sum_j \left(\sum_i a_{ki} b_{ij} \right) u_j \right] \mathbf{f}_k = \sum_k \left[\sum_j (\underline{\underline{AB}})_{kj} u_j \right] \mathbf{f}_k , \end{aligned}$$

therefore the k th component of $(\mathbf{AB})\mathbf{u}$ is

$$[(\mathbf{AB})\mathbf{u}]_k = \sum_j (\underline{\underline{AB}})_{kj} u_j .$$

This together with the result from part (a) allows us to conclude that the matrix of the linear operator \mathbf{AB} is $\underline{\underline{AB}}$.

- (c) Let $\mathbf{C} : V \rightarrow V$ be an *invertible* linear operator, i.e., a linear operator that has an inverse \mathbf{C}^{-1} , so that $\mathbf{C}(\mathbf{C}^{-1}) = \mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$ (where \mathbf{I} is the identity operator: $\mathbf{I}\mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$, the matrix of \mathbf{I} in any basis is the unit matrix $\underline{\underline{I}} = (\delta_{ij})$). Let the operators \mathbf{C} and \mathbf{C}^{-1} have matrices $\underline{\underline{C}} = (c_{ij})$ and $\underline{\underline{D}} = (d_{ij})$, in the basis $\mathbf{f}_1, \dots, \mathbf{f}_n$:

$$\mathbf{C}\mathbf{f}_j = \sum_{i=1}^n c_{ij} \mathbf{f}_i , \quad \mathbf{C}^{-1}\mathbf{f}_j = \sum_{i=1}^n d_{ij} \mathbf{f}_i .$$

Show that the matrix $\underline{\underline{D}}$ corresponding to \mathbf{C}^{-1} is equal to the inverse of the matrix $\underline{\underline{C}}$ corresponding to \mathbf{C} , i.e., $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$.

Solution: By the result of part (b), the operator $\mathbf{C}(\mathbf{C}^{-1})$ has matrix $\underline{\underline{C}}\underline{\underline{D}}$, so $\mathbf{C}(\mathbf{C}^{-1}) = \mathbf{I}$ implies that $\underline{\underline{C}}\underline{\underline{D}} = \underline{\underline{I}}$, i.e., that $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$.

- (d) Let us use the operators \mathbf{C} and \mathbf{C}^{-1} to define a new basis in V . Let the vectors of the new basis $\tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_n$ be defined by

$$\mathbf{f}_j = \mathbf{C} \tilde{\mathbf{f}}_j = \sum_{i=1}^n c_{ij} \tilde{\mathbf{f}}_i ;$$

clearly, this implies that

$$\tilde{\mathbf{f}}_j = \mathbf{C}^{-1} \mathbf{f}_j = \sum_{i=1}^n d_{ij} \mathbf{f}_i \left(= \sum_{i=1}^n (\underline{\underline{C}}^{-1})_{ij} \mathbf{f}_i \right) .$$

Prove that if \tilde{u}_j are the components of \mathbf{u} in the new basis, i.e., $\mathbf{u} = \sum_{j=1}^n \tilde{u}_j \tilde{\mathbf{f}}_j$, then $\tilde{u}_i = \sum_{j=1}^n c_{ij} u_j$.

Solution:

$$\mathbf{u} = \sum_j u_j \mathbf{f}_j = \sum_j u_j \mathbf{C} \tilde{\mathbf{f}}_j = \sum_j u_j \sum_i c_{ij} \tilde{\mathbf{f}}_i = \sum_i \left(\sum_j c_{ij} u_j \right) \tilde{\mathbf{f}}_i$$

– compare with

$$\mathbf{u} = \sum_j \tilde{u}_j \tilde{\mathbf{f}}_j$$

to get

$$\tilde{u}_i = \sum_j c_{ij} u_j .$$

- (e) Now we want to find the matrix of the operator \mathbf{A} in the new basis $\tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_n$. Define the matrix elements \tilde{a}_{ij} of \mathbf{A} in the new basis by

$$\mathbf{A} \tilde{\mathbf{f}}_j =: \sum_{i=1}^n \tilde{a}_{ij} \tilde{\mathbf{f}}_i ,$$

and let $\tilde{\underline{\underline{A}}} = (\tilde{a}_{ij})$. Show that $\tilde{\underline{\underline{A}}} = \underline{\underline{C}} \underline{\underline{A}} \underline{\underline{C}}^{-1}$.

Solution: Using the formulas from part (d) that connect the bases \mathbf{f}_i and $\tilde{\mathbf{f}}_i$, we obtain

$$\begin{aligned} \mathbf{A} \tilde{\mathbf{f}}_j &= \mathbf{A} \sum_i d_{ij} \tilde{\mathbf{f}}_i = \sum_i d_{ij} \mathbf{A} \tilde{\mathbf{f}}_i \\ &= \sum_{i,k} d_{ij} a_{ki} \tilde{\mathbf{f}}_k = \sum_{i,k,l} d_{ij} a_{ki} c_{lk} \tilde{\mathbf{f}}_l = \sum_l \left(\sum_{i,k} c_{lk} a_{ki} d_{ij} \right) \tilde{\mathbf{f}}_l . \end{aligned}$$

Comparing this with $\mathbf{A} \tilde{\mathbf{f}}_j = \sum_l \tilde{a}_{lj} \tilde{\mathbf{f}}_l$, we see that

$$\tilde{a}_{lj} = \sum_{i,k} c_{lk} a_{ki} d_{ij} ,$$

i.e., that $\tilde{\underline{\underline{A}}} = \underline{\underline{C}} \underline{\underline{A}} \underline{\underline{C}}^{-1}$ (recall that $\underline{\underline{C}}^{-1} = (d_{ij})$).