MATH 3423 Homework 8 Due Fri, 11/8/13

Problem 1. Determine the geometric meaning of the operators A, B, and C acting on $\mathbb{R}^2$, if they are represented by the following matrices:

$$\underline{\underline{A}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \underline{\underline{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Hint: Take an arbitrary vector in $\mathbb{R}^2$, say $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, draw $\mathbf{u}$ and at the products $\underline{\underline{A}}\mathbf{u}$, $\underline{\underline{B}}\mathbf{u}$, and $\underline{\underline{C}}\mathbf{u}$ in $\mathbb{R}^2$, and the geometric meaning of the corresponding operators will be transparent.

Problem 2. Suppose that the linear operator $\mathsf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ transforms $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ into $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ into $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ into $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.
Find the matrix $\underline{\underline{A}}$ that corresponds to the operator A.

Hint: You may use that, if $\mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the standard basis in $\mathbb{R}^2$, then $\mathbf{u}_1 = \mathbf{f}_1 + \mathbf{f}_3$, $\mathsf{A}\mathbf{u}_1 = \mathbf{v}_1 = 2\mathbf{f}_1 + \mathbf{f}_2 - \mathbf{f}_3$, etc.

Problem 3.

(a) Prove that $(\underline{\underline{AB}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$.

(b) Directly from the definition of orthogonality of matrices (i.e., $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$), prove that the product of two orthogonal matrices is orthogonal.

Problem 4. Determine the eigenvalues and eigenvectors of the matrix $\underline{\underline{A}} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. How many linearly independent eigenvectors does it have?

Remark: This problem shows the trouble one may encounter in the case of repeated eigenvalues.

Problem 5. Express the coefficients of the characteristic polynomial of the matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of $\det A$ and $\operatorname{tr} A$.

Problem 6. Consider the linear constant coefficient system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 \\ \dot{x}_2 &= 2x_1 + x_2 \ . \end{aligned} \tag{1}$$

(a) Write the system (1) in the form $\dot{\mathbf{x}} = \underline{\underline{A}}\mathbf{x}$. Note that $\underline{\underline{A}}$ is a symmetric matrix.

(b) What does the general theory claim about the eigenvalues and eigenvectors of the matrix $\underline{\underline{A}}$?

(c) Find the eigenvectors and the normalized eigenvectors of the symmetric matrix $\underline{\underline{A}}$.

(d) Show that the eigenvectors and eigenvectors of $\underline{\underline{A}}$ found in (c) satisfy the properties that you predicted in (b).

(e) Write down the matrix $\underline{\underline{S}}$ whose columns are the normalized eigenvectors of $\underline{\underline{A}}$.

(f) Find $\underline{\underline{S}}^{-1}$. You do not need to do any calculations, but please explain what properties you are using.

(g) Find $\underline{\underline{D}} = \underline{\underline{S}}^{-1}\underline{\underline{AS}}$ and compute $\mathrm{e}^{\underline{\underline{D}}t}$.

(h) Use your results from parts (e)–(g) to compute $\mathrm{e}^{\underline{\underline{A}}t}$.

(i) Use your result from part (h) to find the solution of the system (1) if $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Problem 7. Solve the linear constant coefficient system (1) from the previous problem with initial condition $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ by using that, if all eigenvalues λ_j of the matrix $\underline{\underline{A}}$ are distinct, then the general solution of the system $\dot{\mathbf{x}} = \underline{\underline{A}}\mathbf{x}$ is given by

$$\mathbf{x}(t) = \sum_{j=1}^{n} C_j \mathrm{e}^{\lambda_j t} \mathbf{u}_j \ ,$$

where $\mathbf{u}_j$ are the corresponding eigenvectors (not necessarily normalized).