MATH 5763 **Homework 7** **Due Tue, 3/26/2019**

Problem 1. [Poisson process defined through the inter-event times]

Recall that in class we showed that a Poisson process $\{N_t\}_{t\geq 0}$ of rate λ can be constructed as follows. Let X_1, X_2, ... be i.i.d. exponential random variables with parameter λ, i.e., $X_m \sim \mathrm{Exp}(\lambda)$; these random variables play the role of the inter-event time intervals (i.e., the time intervals between two consecutive events). Then the time of the nth event is $T_n = \sum_{m=1}^{n} X_m$, and the Poisson process $\{N_t\}$ can be obtained by $N_t = \max\{n \in \mathbb{Z}_+ : T_n \leq t\}$. We proved that the time T_n of the nth event is a $\Gamma(n, \lambda)$ random variable. It is not difficult to prove by induction that the c.d.f. of $T_n \sim \Gamma(n, \lambda)$ is

$$F_{\Gamma(n,\lambda)}(t) = \begin{cases} 0 \ , & \text{if } t < 0 \ , \\ 1 - \mathrm{e}^{-\lambda t} \displaystyle\sum_{m=0}^{n-1} \frac{(\lambda t)^m}{m!} \ , & \text{if } t \geq 0 \ . \end{cases}$$

In this problem you will relate the Poisson process $\{N_t\}$ with the Γ random variables in a way different from the one we used in class.

(a) Explain (in words) why $N_t = n$ is equivalent to $T_n \leq t < T_{n+1}$.

(b) For any value of $t \geq 0$, how are the events $\{T_{n+1} \leq t\}$ and $\{T_n \leq t\}$ related?

(c) For any value of $t \geq 0$, express the event $\{N_t = n\}$ in terms of the events $\{T_{n+1} \leq t\}$ and $\{T_n \leq t\}$.

(d) Assume that you know that $T_n \sim \Gamma(n, \lambda)$ and also know the explicit expression for $F_{\Gamma(n,\lambda)}$ (given above), and use your result from part (c) to find $\mathbb{P}(N_t = n)$.

Problem 2. [Poisson vs. non-Poisson process]

A system is made up of two components. We suppose that the lifetime (in years) of each component has an exponential distribution with parameter $\lambda = 2\,\mathrm{year}^{-1}$, and that the components operate independently. When the system goes down, the two components are then immediately replaced by new ones. Consider the following three cases:

I. the two components are placed in series (so that both components must function for the system to work);

II. the two components are placed in parallel (so that a single operating component is sufficient for the system to function) and the two components operate at the same time;

III. the two components are placed in parallel, but only one component operates at a time, and the other component is in standby (i.e., ready to replace the first component when it fails).

Let $\{N_t : t \geq 0\}$ be the number of system failures in the interval $[0, t]$. Answer the following questions in each of the cases above.

(a) Is $\{N_t : t \geq 0\}$ a Poisson process? If it is, what is its rate? If it is not, justify, and determine the probability distribution of the inter-event times τ_j.

(b) What is the average time elapsed between two consecutive system failures? In two of the above three cases the answer is obvious (but I do want to see your calculations). Please discuss your results in these two cases.

Remark: This problem is related to many facts that you have already seen – think, e.g., what are the connections with Problems 3 and 4 of Homework 1 and with Problem 5(a) of Homework 5.

Problem 3. [Homogeneous vs. non-homogeneous Poisson processes]

Let $\{N_t : t \geq 0\}$ be a Poisson process with constant rate λ. Define the continuous-time processes $\{K_t : t \geq 0\}$, $\{L_t : t \geq 0\}$, $\{M_t : t \geq 0\}$ by

$$K_t = N_{t+2} - N_2 \ , \qquad L_t = N_{2t} \ , \qquad M_t = N_{\sqrt{t}} \ .$$

Which of these processes is/are also a constant-rate Poisson process? What are their rates? Justify your answers briefly.

Hint: It is easy to find the distributions of the random variables K_t, L_t, and M_t, since you know that N_t has Poisson distribution with parameter λt.

Problem 4. [Probability generating functions of compound random variables]

The *(probability) generating function* of a random variable X taking values in $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ is defined to be the generating function of its probability mass function p_X:

$$G_X(s) = \mathbb{E}[s^X] = \sum_{n=0}^{\infty} p_X(n)\, s^n \quad \left(= \sum_{n=0}^{\infty} \mathbb{P}(X = n)\, s^n \right) \ .$$

(a) Let X_1, X_2, ... be a sequence of independent identically distributed random variables (with values in $\mathbb{Z}_+$) with common generating function G_X. Let N be a random variable taking values in $\mathbb{Z}_+$ which is independent of the X_j's, and let G_N be the generating function of N. Define the random variable Y_N by $Y_N = X_1 + X_2 + \cdots + X_N$ if $N \geq 1$ and $Y_N = 0$ if $N = 0$. Show that Y_N has generating function given by $G_{Y_N}(s) = G_N(G_X(s))$.

(b) Let Z be a Poisson random variable with parameter Λ, where Λ is a Poisson random variable with parameter μ. Compute G_Z and $\mathbb{E}[Z]$.

Hint: The generating function of $\Lambda \sim \text{Poisson}(\mu)$ is $G_\Lambda(s) = \mathrm{e}^{\mu(s-1)}$.

(c) Let V be a Poisson random variable with parameter Θ, where Θ is an exponential random variable with parameter ν. Show that $V + 1$ has a geometric distribution, and find the parameter of this distribution by using two methods:

– direct computation of the p.m.f. of $V + 1$:

$$p_{V+1}(n) = \mathbb{P}(V + 1 = n) = \mathbb{P}(V = n - 1) = \mathbb{E}\big[\mathbb{P}(V = n - 1 | \Theta)\big] \ ;$$

– computing the generating function G_V or G_{V+1} (and then convincing me that your result indeed implies the desired conclusion).

Hint: If $\Theta \sim \mathrm{Exp}(\nu)$, then $f_\Theta(x) = \nu \mathrm{e}^{-\nu x} \chi_{[0,\infty)}(x)$; if $W \sim \mathrm{Geom}(p)$, then $p_W(n) = (1-p)^{n-1} p$ for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and $G_W(s) = \dfrac{ps}{1-(1-p)s}$. Also, recall that $\Gamma(n) = (n-1)!$.

Food for Thought Problem 1. The toll collected from the traffic passing through the toll booth on Highway 44 between Oklahoma City and Tulsa can be modeled for the hours between 9 a.m. and 5 p.m. by a compound Poisson process. Assume that the toll booth serves the arriving vehicles instantaneously, so that there are no waiting lines.

The vehicles can be divided into two big categories – personal vehicles and commercial vehicles. The personal vehicles arrive at the toll booth with average frequency 7 personal vehicles per minute, while the commercial vehicles arrive with average frequency 3 commercial vehicles per minute.

There are three types of personal vehicles – 80 % of the personal vehicles are cars, 15 % are SUVs and 5 % are RVs.

There are four types of commercial vehicles – pick-up trucks, normal-size trucks, 18-wheelers, and busses; the probability with which a commercial vehicle belongs to each of these four types is 40 %, 30 %, 20 %, and 10 %, respectively.

The toll rates are the following: car \$1, SUV \$3, RV \$5; pick-up truck \$3, normal-size truck \$5, 18-wheeler \$8, bus \$10.

Please answer the questions below. Define clearly your notations, and use the concrete numbers given in this problem. You are allowed to use the theoretical results derived in class, but please write explicitly what results you use.

(a) Think of the toll collected from the personal vehicles as a compound Poisson process $Y_1(t)$, where at each arrival of a personal vehicle the collected toll is random. What is the p.m.f. of the random variable describing the collected toll from a personal vehicle? What is the rate of the Poisson process describing the moments of arrival of personal vehicles?

(b) Find the moment generating function of the process $Y_1(t)$, and use your result to find the average value of the toll collected in a period of 1 hour, and the variance of this toll.

(c) Answer the same questions as in part (a) about the Poisson process $Y_2(t)$ describing the toll collected from the commercial vehicles.

(d) Answer the same questions as in (b), but for the process $Y_2(t)$.

(e) Define the random process $Y(t) = Y_1(t) + Y_2(t)$ of the toll collected from all vehicles passing through the toll booth. We can think of this random process as a compound Poisson process. What is the frequency of the Poisson process with which the events of this random process occur? What is the p.m.f. of the toll collected from each vehicle passing through the toll booth (without making a distinction between personal and commercial vehicles)?

(f) Write explicitly the moment generating function of the random process $Y(t)$, as well as $\mathbb{E}[Y(t)]$ and the variance of $Y(t)$. On average, how much toll will be collected from 10 a.m. to 11 a.m.?