

Problem 1. Let $\{N_t : t \geq 0\}$ be a Poisson process with constant rate λ . Define the continuous-time processes $\{K_t : t \geq 0\}$, $\{L_t : t \geq 0\}$, $\{M_t : t \geq 0\}$ by

$$K_t = N_{t+2} - N_2, \quad L_t = N_{2t}, \quad M_t = N_{\sqrt{t}}.$$

Which of these processes is/are also a constant-rate Poisson process? What are their rates? Justify your answers briefly.

Hint: It is easy to find the distributions of the random variables K_t , L_t , and M_t , since you know that N_t has Poisson distribution with parameter λt .

Problem 2. A system is made up of two components. We suppose that the lifetime (in years) of each component has an exponential distribution with parameter $\lambda = 2\text{year}^{-1}$, and that the components operate independently. When the system goes down, the two components are then immediately replaced by new ones. Consider the following three cases:

- I. the two components are placed in series (so that both components must function for the system to work);
- II. the two components are placed in parallel (so that a single operating component is sufficient for the system to function) and the two components operate at the same time);
- III. the two components are placed in parallel, but only one component operates at a time, and the other component is in standby (i.e., ready to replace the first component when it fails).

Let $\{N_t : t \geq 0\}$ be the number of system failures in the interval $[0, t]$. Answer the following questions in each of the cases above.

- (a) Is $\{N_t : t \geq 0\}$ a Poisson process? If it is, what is its rate? If it is not, justify, and determine the probability distribution of the inter-event times τ_j .
- (b) What is the average time elapsed between two consecutive system failures? In two of the above three cases the answer is obvious (but I do want to see your calculations). Please discuss your results in these two cases.

Problem 3. The toll collected from the traffic passing through the toll booth on Highway 44 between Oklahoma City and Tulsa can be modeled for the hours between 9 a.m. and 5 p.m. by a compound Poisson process. Assume that the toll booth serves the arriving vehicles instantaneously, so that there are no waiting lines.

The vehicles can be divided into two big categories – personal vehicles and commercial vehicles. The personal vehicles arrive at the toll booth with average frequency 7 personal vehicles per minute, while the commercial vehicles arrive with average frequency 3 commercial vehicles per minute.

There are three types of personal vehicles – 80 % of the personal vehicles are cars, 15 % are SUVs and 5 % are RVs.

There are four types of commercial vehicles – pick-up trucks, normal-size trucks, 18-wheelers, and busses; the probability with which a commercial vehicle belongs to each of these four types is 40 %, 30 %, 20 %, and 10 %, respectively.

The toll rates are the following: car \$1, SUV \$3, RV \$5; pick-up truck \$3, normal-size truck \$5, 18-wheeler \$8, bus \$10.

Please answer the questions below. Define clearly your notations, and use the concrete numbers given in this problem. You are allowed to use the theoretical results derived in class, but please write explicitly what results you use.

- (a) Think of the toll collected from the personal vehicles as a compound Poisson process $Y_1(t)$, where at each arrival of a personal vehicle the collected toll is random. What is the p.m.f. of the random variable describing the collected toll from a personal vehicle? What is the rate of the Poisson process describing the moments of arrival of personal vehicles?
- (b) Find the moment generating function of the process $Y_1(t)$, and use your result to find the average value of the toll collected in a period of 1 hour, and the variance of this toll.
- (c) Answer the same questions as in part (a) about the Poisson process $Y_2(t)$ describing the toll collected from the commercial vehicles.
- (d) Answer the same questions as in (b), but for the process $Y_2(t)$.
- (e) Define the random process $Y(t) = Y_1(t) + Y_2(t)$ of the toll collected from all vehicles passing through the toll booth. We can think of this random process as a compound Poisson process. What is the frequency of the Poisson process with which the events of this random process occur? What is the p.m.f. of the toll collected from each vehicle passing through the toll booth (without making a distinction between personal and commercial vehicles)?
- (f) Write explicitly the moment generating function of the random process $Y(t)$, as well as $\mathbb{E}[Y(t)]$ and the variance of $Y(t)$. On average, how much toll will be collected from 10 a.m. to 11 a.m.?

Food for Thought Problem 1. A sequence $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ of real numbers can contain a lot of information. One concise way of storing this information is to wrap up the numbers a_n together in a “generating function”. For example, let us define the (ordinary) *generating function* of the sequence \mathbf{a} in the function $G_{\mathbf{a}}$ defined by

$$G_{\mathbf{a}}(s) = \sum_{n=0}^{\infty} a_n s^n \quad \text{for those } s \in \mathbb{R} \text{ for which the sum converges.}$$

The sequence \mathbf{a} may be reconstructed from the function $G_{\mathbf{a}}$ by setting $a_n = \frac{1}{n!} G_{\mathbf{a}}^{(n)}(0)$, where $f^{(n)}$ denotes the n th derivative of the function f . Generating functions are considered in many books

on combinatorics, discrete mathematics, and probability (among others); a readable and freely available book is Herbert Wilf's *generatingfunctionology* (its second edition is freely available at www.math.upenn.edu/~wilf/DownldGF.html).

The *convolution* of the sequences $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ and $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$ is the sequence $\mathbf{c} = \{c_n\}_{n=0}^{\infty}$ defined by

$$c_n = \sum_{k=0}^n a_k b_{n-k} \quad \left(= \sum_{k=0}^n a_{n-k} b_k \right) .$$

Sometimes the convolution of \mathbf{a} and \mathbf{b} is denoted by $\mathbf{a} * \mathbf{b}$.

- (a) Let $z_n = (\cos \theta + i \sin \theta)^n$, where $i = \sqrt{-1}$, and θ is a fixed real number. Show that the generating function of the sequence $\mathbf{z} = \{z_n\}_{n=0}^{\infty}$ is

$$G_{\mathbf{z}}(s) = \frac{1}{1 - s(\cos \theta + i \sin \theta)} \quad \text{for } |s| < 1 .$$

- (b) Prove that, if \mathbf{a} and \mathbf{b} have generating functions $G_{\mathbf{a}}$ and $G_{\mathbf{b}}$, then the generating function of $\mathbf{c} = \mathbf{a} * \mathbf{b}$ is $G_{\mathbf{c}}(s) = G_{\mathbf{a}}(s) G_{\mathbf{b}}(s)$.

- (c) Obtain the combinatorial identity $\sum_{k=0}^N \binom{N}{k}^2 = \binom{2N}{N}$, where $N \in \mathbb{N}$, by noticing that its left-hand side can be thought of as the convolution of the sequence

$$a_n = \begin{cases} \binom{N}{n} & \text{for } 0 \leq n \leq N \\ 0 & \text{for } n \geq N+1 \end{cases}$$

with itself, and using the fact proved in part (b) about $G_{\mathbf{a} * \mathbf{a}}$.

- (d) It is a well-known fact from elementary probability (and an easy exercise for you) that if X and Y are independent random variables taking values in $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$, then the probability mass function of their sum, $Z = X + Y$, is the convolution of the probability mass functions of X and Y : $p_Z = p_X * p_Y$. Here we think of a p.m.f. p_X as a sequence \mathbf{a} with $a_n = p_X(n) = \mathbb{P}(X = n)$.

Let X and Y be independent Poisson random variables with parameters λ and μ , respectively. Compute explicitly the probability generating functions

$$G_X(s) = \sum_{n=0}^{\infty} p_X(n) s^n \quad \text{and} \quad G_Y(s) = \sum_{n=0}^{\infty} p_Y(n) s^n ,$$

and use them to show that $X + Y$ is a Poisson random variable and to find the parameter of the distribution of $X + Y$.

Food for Thought Problem 2. Let the function $F : \mathbb{R} \rightarrow [0, 1]$ shown in Figure 1 be defined by $F(x) = 0$ for $x \leq 0$, $F(x) = 1$ for $x \geq 1$, and

$$F(x) = \frac{1}{2^{j-1}} \quad \text{for} \quad \frac{1}{2^j} \leq x < \frac{1}{2^{j-1}} , \quad j \in \{1, 2, 3, \dots\}$$

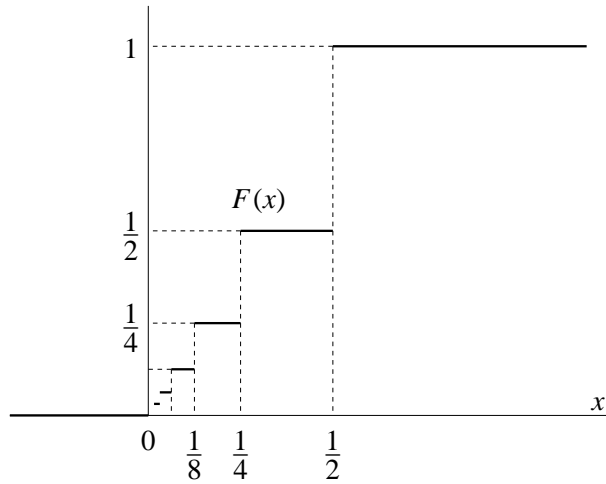


Figure 1: Graph of the function F .

Show that $\int_{\mathbb{R}} dF(x) = 1$, $\int_{\mathbb{R}} x dF(x) = \frac{1}{3}$, $\int_{\mathbb{R}} \ln x dF(x) = -2 \ln 2$.

Hint: You will need that, for $|q| < 1$, $\sum_{k=1}^{\infty} k q^{k-1} = \frac{d}{dq} \sum_{k=0}^{\infty} q^k = \frac{d}{dq} \frac{1}{1-q} = \frac{1}{(1-q)^2}$.