MATH 5483 Homework 7 Due Thu, Mar 6

Problem 1. In this problem you will prove that, for any positive constant Ω, the functions

$$g_n(x) := \frac{\sin\left(\pi\Omega\left(x - \frac{n}{\Omega}\right)\right)}{\pi\sqrt{\Omega}\left(x - \frac{n}{\Omega}\right)} , \qquad n \in \mathbb{Z} ,$$

form an orthonormal system on $\mathbb{R}$. Please follow the steps below.

(a) Let $\widehat{g}(\gamma) := \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$. Show that

$$g(x) = \frac{\sin(\pi x)}{\pi x} .$$

Remark: Since $g(x)$ is in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$, one needs to define the Fourier transform carefully. There are theorems that guarantee that all this makes sense for $g \in L^2(\mathbb{R})$ (see Section 3.6 of the book), so you do not need to worry about these subtleties here – just do the calculation formally.

(b) Show in detail that

$$g_n(x) = (D_\Omega T_n g)(x) ,$$

where D_a and T_b are the dilation and translation operators defined as usual (see Definition 3.39 on pages 79–80).

(c) Use the properties of the Fourier transform from Theorem 3.48 and Parseval's formula (Theorem 3.27; see also Theorem 3.32) to show that $\{g_n\}_{n\in\mathbb{Z}}$ form an orthonormal system on $\mathbb{R}$.

Problem 2.

(a) Prove directly from the definition that, if $f_{jk}(x) = (D_{2^j} T_k f)(x) = 2^{j/2} f(2^j x - k)$, then

$$\widehat{f_{jk}}(\gamma) = 2^{-j/2} \mathrm{e}^{-2\pi i k\gamma/2^j} \, \widehat{f}\left(\frac{\gamma}{2^j}\right) .$$

(b) Use your result from (a) to show that if the function $f(x)$ satisfies

$$f(x) = \frac{1}{2} f(2x+1) + f(2x) + \frac{1}{2} f(2x-1) ,$$

then its Fourier transform satisfies

$$\widehat{f}(\gamma) = \cos^2\left(\frac{\pi\gamma}{2}\right) \widehat{f}\left(\frac{\gamma}{2}\right) .$$

Problem 3. In this problem you will prove some properties of signals and systems. We have not studied this material in the lectures, but it is very easy. Pages 87–92 of the book contain all definitions and facts that you need to solve this problem.

(a) Let $|a| < 1$, $N \in \mathbb{N}$, and let the signal $x(n)$ be defined by

$$x(n) = \begin{cases} a^n & \text{if } |n| \leq N \ , \\ 0 & \text{if } |n| > N \ . \end{cases}$$

Show that the frequency domain representation $\widehat{x}(\omega)$ of the signal $x(n)$ is

$$\widehat{x}(\omega) = \frac{1 - \left(a\, \mathrm{e}^{-2\pi i\omega}\right)^{N+1}}{\left(1 - a\, \mathrm{e}^{-2\pi i\omega}\right)^{N+1}} \ ,$$

and find the z-transform $X(z)$ of $x(n)$.

(b) Use some trigonometric identities to find the signal $x(n)$ whose frequency domain representation is

$$\widehat{x}(\gamma) = 7 + 4\cos^2(3\gamma) \ .$$

(c) What can you say about the signal $x(n)$ if you know that its frequency domain representation is a real-valued function?

(d) Let $\delta(n)$ be the unit impulse signal (defined as $\delta(n) := \delta_{n0}$, where δ_{nm} is the Kronecker's symbol), and τ_m be the translation operator, $(\tau_m x)(n) := x(n - m)$. Let $(x * y)(n)$ stand for the convolution of the signals $x(n)$ and $y(n)$. Find $(x * (\tau_m \delta))(n)$.

(e) Compute the frequency domain representations $\widehat{\tau_m \delta}(\omega)$ and

$$\sum_{m=-M}^{M} \widehat{\tau_m \delta}(\omega) \ .$$

(f) Directly from the definition of convolution of signals, prove that

$$\widehat{x * y}(\omega) = \widehat{x}(\omega)\, \widehat{y}(\omega) \ .$$

(g) Let the signal $x(n)$ be N-periodic, i.e., $x(n + N) = x(n)$ for any $n \in \mathbb{Z}$. Show that for any signal $h(n)$, the convolution $(x * h)(n)$ is also N-periodic.