MATH 5453 Homework 7 Due Thu, Oct 23

Problem 17 from Section 2.2 of the book.

Hint: The fact that the sequence $\{f_n\}$ is increasing implies that $f_n \le f := \lim_{k\to\infty} f_k$, which, together with the monotonicity of the integral, shows that $\int f_n \le \int f$ for each $n \in \mathbb{N}$, and taking the limit $n \to \infty$ implies $\lim_{n\to\infty} \displaystyle\int f_n \le \int f$. The other inequality, $\lim_{n\to\infty} \displaystyle\int f_n \ge \int f$, follows easily from Fatou's Lemma.

Problems 18, 22 from Section 2.3 of the book.

Additional problem 1. Let $(X, \mathcal{M})$ be a measurable space, and $A \subset X$. Prove that the collection of sets

$$\mathcal{M}_A := \{A \cap E : E \in \mathcal{M}\}$$

is a σ-algebra of subsets of A.

Hint: "Complement" of a set G from $\mathcal{M}_A$ means "complement of the set in A", i.e., $A \setminus G$.

Additional problem 2. Describe the σ-algebra $\mathcal{M}$ on $\mathbb{R}$ such that all even functions $f : \mathbb{R} \to \mathbb{R}$ are $(\mathcal{M}, \mathcal{B})$-measurable, and all non-even functions $f : \mathbb{R} \to \mathbb{R}$ are not $(\mathcal{M}, \mathcal{B})$-measurable.

Hint: Can the set $(-3, -2) \cup (1, 5)$ possibly be an element of $\mathcal{M}$?

Additional problem 3. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = x\chi_{[0,\frac{1}{2}]}(x) + \chi_{(\frac{1}{2},1]}(x) \ .$$

Describe explicitly the smallest σ-algebra $\mathcal{M}$ of subsets of $[0, 1]$ such that the function f is $(\mathcal{M}, \mathcal{B})$-measurable.

Hint: This is an easy problem – just think of all possible preimages of open intervals (a, b), $a < b$ that generage $\mathcal{B}$. You have to consider all possible cases separately: $1 < a < b$, $\frac{1}{2} < a < 1 < b$, $0 < a < \frac{1}{2} < 1 < b$, $a < 0 < 1 < b$, $\frac{1}{2} < a < b < 1$, $0 < a < \frac{1}{2} < b < 1$, $a < 0 < \frac{1}{2} < b < 1$, $0 < a < b < \frac{1}{2}$, $a < 0 < b < \frac{1}{2}$, $a < b < 0$. Having done this, try to describe your observations simply.

Additional problem 4. Find the limit

$$\lim_{n\to\infty} \int_{[-20,20]} \sin^4(x^3)\, \mathrm{e}^{-nx^2}\, \mathrm{d}m(x)$$

without computing the integral explicitly. You have to do this using:

(a) the Monotone Convergence Theorem (more precisely, its version for decreasing sequences of functions which was Problem 1(b) of Homework 6);

(b) the Dominated Convergence Theorem.

Please check explicitly if all the conditions in the theorems are satisfied.

Remark: The fact that the integrand above is a measurable function follows immediately from the fact that it is continuous. (Why? What do you know about the preimages of open sets under continuous functions?)

Food for thought.[1] Let $(X, \mathcal{M}, \mu)$ be a measure space. For $f \in L^+$, define the map $\lambda : \mathcal{M} \to [0, \infty]$ by

$$\lambda(E) := \int_E f \, \mathrm{d}\mu \; .$$

Show that λ is a measure on $\mathcal{M}$, and for any $g \in L^+$,

$$\int g \, \mathrm{d}\lambda = \int f g \, \mathrm{d}\mu \; .$$

Hint: First suppose that g is simple.

[1] "Food for thought" problems are not to be turned in. They are just for you to learn some facts and think about their proofs.