

**Problem 17** from Section 2.2 of the book.

*Hint:* The fact that the sequence  $\{f_n\}$  is increasing implies that  $f_n \leq f := \lim_{k \rightarrow \infty} f_k$ , which, together with the monotonicity of the integral, shows that  $\int f_n \leq \int f$  for each  $n \in \mathbb{N}$ , and taking the limit  $n \rightarrow \infty$  implies  $\lim_{n \rightarrow \infty} \int f_n \leq \int f$ . The other inequality,  $\lim_{n \rightarrow \infty} \int f_n \geq \int f$ , follows easily from Fatou's Lemma.

**Problems 18, 22** from Section 2.3 of the book.

**Additional problem 1.** Let  $(X, \mathcal{M})$  be a measurable space, and  $A \subset X$ . Prove that the collection of sets

$$\mathcal{M}_A := \{A \cap E : E \in \mathcal{M}\}$$

is a  $\sigma$ -algebra of subsets of  $A$ .

*Hint:* "Complement" of a set  $G$  from  $\mathcal{M}_A$  means "complement of the set in  $A$ ", i.e.,  $A \setminus G$ .

**Additional problem 2.** Describe the  $\sigma$ -algebra  $\mathcal{M}$  on  $\mathbb{R}$  such that all even functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  are  $(\mathcal{M}, \mathcal{B})$ -measurable, and all non-even functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  are not  $(\mathcal{M}, \mathcal{B})$ -measurable.

*Hint:* Can the set  $(-3, -2) \cup (1, 5)$  possibly be an element of  $\mathcal{M}$ ?

**Additional problem 3.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = x\chi_{[0, \frac{1}{2}]}(x) + \chi_{(\frac{1}{2}, 1]}(x) .$$

Describe explicitly the smallest  $\sigma$ -algebra  $\mathcal{M}$  of subsets of  $[0, 1]$  such that the function  $f$  is  $(\mathcal{M}, \mathcal{B})$ -measurable.

*Hint:* This is an easy problem – just think of all possible preimages of open intervals  $(a, b)$ ,  $a < b$  that generate  $\mathcal{B}$ . You have to consider all possible cases separately:  $1 < a < b$ ,  $\frac{1}{2} < a < 1 < b$ ,  $0 < a < \frac{1}{2} < 1 < b$ ,  $a < 0 < 1 < b$ ,  $\frac{1}{2} < a < b < 1$ ,  $0 < a < \frac{1}{2} < b < 1$ ,  $a < 0 < \frac{1}{2} < b < 1$ ,  $0 < a < b < \frac{1}{2}$ ,  $a < 0 < b < \frac{1}{2}$ ,  $a < b < 0$ . Having done this, try to describe your observations simply.

**Additional problem 4.** Find the limit

$$\lim_{n \rightarrow \infty} \int_{[-20, 20]} \sin^4(x^3) e^{-nx^2} dm(x)$$

without computing the integral explicitly. You have to do this using:

- (a) the Monotone Convergence Theorem (more precisely, its version for decreasing sequences of functions which was Problem 1(b) of Homework 6);
- (b) the Dominated Convergence Theorem.

Please check explicitly if all the conditions in the theorems are satisfied.

*Remark:* The fact that the integrand above is a measurable function follows immediately from the fact that it is continuous. (Why? What do you know about the preimages of open sets under continuous functions?)

**Food for thought.**<sup>1</sup> Let  $(X, \mathcal{M}, \mu)$  be a measure space. For  $f \in L^+$ , define the map  $\lambda : \mathcal{M} \rightarrow [0, \infty]$  by

$$\lambda(E) := \int_E f \, d\mu .$$

Show that  $\lambda$  is a measure on  $\mathcal{M}$ , and for any  $g \in L^+$ ,

$$\int g \, d\lambda = \int fg \, d\mu .$$

*Hint:* First suppose that  $g$  is simple.

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<sup>1</sup>“Food for thought” problems are not to be turned in. They are just for you to learn some facts and think about their proofs.