MATH 4163 **Homework 7** **Due Thu, 03/15/2018**

Problem 1. [The heat equation in a cube]

Consider the heat equation for the temperature $u = u(x, y, z, t)$ in a cube, with Dirichlet BCs on some walls, and Neumann BCs on others. Use the method of separation of variables to solve it. You may use the results for the eigenfunctions of the BVPs for ODEs that we have considered in class (or recall Problem 2 of Homework 4), but please write clearly what results you are using. In particular, you can write right away the eigenvalues and the eigenfunctions, without deriving them. The BCs correspond to maintaining zero temperature on the side walls, while keeping the floor and the ceiling thermally insulated. Before the final step (imposing the initial conditions), write the general form of the solution obtained by separating variables (with arbitrary constants).

$$\begin{aligned}
&u_t = 9\,\Delta u\ , \qquad (x, y, z, t) \in [0, \pi] \times [0, \pi] \times [0, \pi] \times \mathbb{R}_+\ ,\\
&u(0, y, z, t) = 0\ , \quad u(\pi, y, z, t) = 0\ ,\\
&u(x, 0, z, t) = 0\ , \quad u(x, \pi, z, t) = 0\ ,\\
&u_z(x, y, 0, t) = 0\ , \quad u_z(x, y, \pi, t) = 0\ ,\\
&u(x, 0) = \sin 2x \sin 5y - 5 \sin x \sin 2y \cos 3z\ .
\end{aligned} \tag{1}$$

Problem 2. [Heat equation with sources of heat]

The method of separation of variables can also be applied when there are sources of heat inside the spatial domain. In this problem you will solve the IBVP

$$\begin{aligned}
&u_t = 4u_{xx} + 7 \sin 3x + \mathrm{e}^{-t} \sin 5x\ , \qquad (x, t) \in [0, \pi] \times \mathbb{R}_+\ ,\\
&u(0, t) = 0\ , \quad u(\pi, t) = 0\ ,\\
&u(x, 0) = 8 \sin 3x\ .
\end{aligned} \tag{2}$$

The term

$$Q(x, t) = 7 \sin 3x + \mathrm{e}^{-t} \sin 5x$$

in the right-hand side of the PDE corresponds to the density of the sources of heat inside the spatial domain $[0, \pi]$.

(a) If there were no sources of heat, then separation of variables will lead to the BVP

$$\begin{aligned}
&X''(x) - \mu X = 0\ , \quad x \in [0, \pi]\ ,\\
&X(0) = 0\ , \quad X(\pi) = 0\ ,
\end{aligned}$$

whose non-trivial solutions are

$$X_n(x) = \sin \lambda_n x = \sin nx\ , \quad \lambda_n = n\ , \quad n = 1, 2, 3, \dots\ ,$$

and the corresponding values of the constant μ from the separation of variables are $\mu_n = -\lambda_n^2 = -n^2$. Therefore, the solution $u(x,t)$ (without source of heat!) would be a superposition of $X_n(x)$ with coefficients $T_n(t)$:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin nx \ . \tag{3}$$

This expression satisfies the BCs in (2), but not the PDE or the initial conditions. Nevertheless, we will try to find a solution fo the IBVP (2) as an expansion of the form (3). We have to find functions $T_n(t)$ to satisfy also the PDE and the initial conditions.

Plug the expression (3) for $u(x,t)$ into the PDE from (2), and use the fact that the functions $X_n(x)$ are linearly independent, to equate the coefficients and write down ODEs that the functions $T_n(t)$ must satisfy. Note that the heat source has the form

$$Q(x,t) = 7\sin 3x + \mathrm{e}^{-t}\sin 5x = \sum_{n=1}^{\infty} Q_n(t) \sin nx \ ,$$

where

$$Q_n(t) = 7\delta_{3n} + \mathrm{e}^{-t}\delta_{5n} \ ,$$

so that the equations for $T_3(t)$ and $T_5(t)$ will look different from the equations for $T_n(t)$ with $n \neq 3, 5$ (only the ODEs with $n = 3, 5$ will have a non-zero right-hand sides).

(b) Set $t = 0$ in (3) and use the initial condition from (2) to find initial conditions for the ODEs for $T_n(t)$ (i.e., find the values of $T_n(0)$).

(c) Write down the ODE for $T_3(t)$ and the initial condition for it.

(d) Solve the initial value problem for the ODE for $T_3(t)$.

(e) Write down the initial value problem for $T_5(t)$.

(f) Solve the initial value problem from part (e), if you know that the general solution of the ODE $y'(t) + ay(t) = b\mathrm{e}^{ct}$ is $y(t) = C\mathrm{e}^{-at} + \frac{b}{a+c}\,\mathrm{e}^{ct}$, where C is an arbitrary constant.

(g) Write down the solution of the IBVP (2).

Problem 3. [Homogenizing time-independent boundary conditions]

So far we always considered homogeneous BCs, like $u(0,t) = 0$ (Dirichlet), $u_x(0,t) = 0$ (Neumann), or $u_x(0,t) + u(0,t) = 0$ (Robin). In this problem you will learn what to do if the BCs are not homogeneous, i.e., the right-hand sides of these expressions are non-zero and do not depend on time.

The following trick usually works for the case when the BCs and the sources of heat do not depend on time, like in the IBVP

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} + Q(x) \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\
&u(0,t) = a \ , \quad u(L,t) = b \ , \\
&u(x,0) = f(x) \ ,
\end{aligned} \tag{4}
$$

where a and b are constants. In this case one looks for the *asymptotic solution* $u_\infty(x)$, i.e., limit of the solution as $t \to \infty$:

$$u_\infty(x) = \lim_{t\to\infty} u(x,t) \ .$$

For $u_\infty(x)$ we obtain the following BVP:

$$
\begin{aligned}
&\alpha^2 u''_\infty(x) + Q(x) = 0 \ , \qquad x \in [0,L] \ , \\
&u_\infty(0) = a \ , \quad u_\infty(L) = b \ .
\end{aligned} \tag{5}
$$

Having found $u_\infty(x)$ that solves the BVP (5), introduce a new unknown function, $v(x,t)$, by

$$v(x,t) = u(x,t) - u_\infty(x) \ .$$

This function will satisfy homogeneous BCs, but the initial condition and the PDE for $v(x,t)$ may differ from the one for $u(x,t)$.

(a) Follow the strategy outlined above to formulate the IBVP for the function $v(x,t)$ if $u(x,t)$ satisfies the following IBVP with Dirichlet BCs:

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\
&u(0,t) = a \ , \quad u(L,t) = b \ , \\
&u(x,0) = f(x) \ ,
\end{aligned} \tag{6}
$$

(where a and b are constants).

(b) Do the same for the following IBVP with one Dirichlet and one Robin BCs:

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\
&u(0,t) = a \ , \quad u_x(L,t) + cu(L,t) = b \ , \\
&u(x,0) = f(x) \ ,
\end{aligned} \tag{7}
$$

where a, b, and c are constants.

(c) Do the same for the following IBVP with one Dirichlet and one Neumann BCs, and with sources of heat in the domain:

$$\begin{aligned} & u_t = \alpha^2 u_{xx} + x \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\ & u(0,t) = a \ , \quad u_x(L,t) = b \ , \\ & u(x,0) = f(x) \ , \end{aligned} \tag{8}$$

where a and b are constants.

Problem 4. [Compatibility between the PDE and the Neumann BCs]
In this problem we will try to follow the strategy used in the previous problem in order to homogenize the BCs in the following IBVP with Neumann BCs and with sources of heat in the domain:

$$\begin{aligned} & u_t = \alpha^2 u_{xx} + Q(x) \ , \qquad (x,t) \in [0,L] \times \mathbb{R}_+ \ , \\ & u_x(0,t) = a \ , \quad u_x(L,t) = b \ , \\ & u(x,0) = f(x) \ , \end{aligned} \tag{9}$$

where a and b are constants, and $Q(x)$ is a given function. Recall that the first step is to find the asymptotic temperature distribution, $u_\infty(x)$.

(a) First try to find the asymptotic temperature distribution $u_\infty(x)$, for the problem (9) with $Q(x) = x$. Can you do it? (Recall that you have no control over the function $Q(x)$, the constants α, a, and b (and L) from the problem.) Describe what you observe.

(b) Integrate the ODE for the stationary temperature distribution $u_\infty(x)$ over x, and formulate a general compatibility condition between the heat sources (described by the function $Q(x)$) and the BCs (given by the constants a and b).

(c) Consider the IBVP

$$\begin{aligned} & u_t = u_{xx} + x \ , \qquad (x,t) \in [0,4] \times \mathbb{R}_+ \ , \\ & u_x(0,t) = 3 \ , \quad u_x(4,t) = -5 \ , \\ & u(x,0) = f(x) \ . \end{aligned} \tag{10}$$

Check that the compatibility condition derived in part (b) is satisfied, find the asymptotic temperature distribution $u_\infty(x)$, and formulate an IBVP for the function $v(x,t) = u(x,t) - u_\infty(x)$.