MATH 4163 Homework 7 Due on Mon, Oct 22, in 4W138

Problem 1. Consider the following BVP for the heat equation on the interval $[0, \pi]$ with time-dependent Dirichlet BCs:

$$\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) \ , \qquad x \in [0,\pi] \ , \quad t > 0 \ , \\
&u(0,t) = 0 \ , \\
&u(\pi,t) = \pi \cos t \ , \\
&u(x,0) = x \ .
\end{aligned}$$

(a) Introduce the reference temperature distribution $r(x,t)$ as suggested on page 351 of the book, in order to transform the BCs to homogeneous ones. What BVP do you obtain for the new unknown function, $v(x,t) = u(x,t) - r(x,t)$? (Do not forget to transform the ICs.)

(b) Write down the system of functions $\{X_n\}$ in which you will expand the unknown function $v(x,t)$:

$$v(x,t) = \sum_n T_n(t)\, X_n(x) \ . \tag{1}$$

Since we have done this many times, just give me the result, no need to derive it.

(c) Substitute the expansion (1) in the BVP for $v(x,t)$ found in part (a) and write down the system of initial value problems (IVPs) for the functions $T_n(t)$.

Hint: You may use that $x \sin t = 2 \sin t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$ for $x \in (0,\pi)$, $t \in \mathbb{R}$.

(d) Write down the general solutions of the ODEs for the functions $T_n(t)$.

Hint: You may use (without deriving it) the fact that the general solution of the ODE $\phi'(t) + n^2\phi(t) = b \sin t$ is

$$\phi(t) = \begin{cases} \dfrac{b}{2}(\sin t - \cos t) + C\mathrm{e}^{-t} & \text{for } n = 1 \ , \\ \dfrac{b}{1+n^4}\,(n^2 \sin t - \cos t) + C\mathrm{e}^{-n^2 t} & \text{for } n = 2, 3, \ldots \ . \end{cases}$$

(e) Impose the initial conditions on the functions $T_n(t)$ from in part (d) in order to obtain the solutions of the IVPs (from part (c)) for the functions $T_n(t)$. (Clearly, you have to do this separately for $n = 1$ and for $n = 2, 3, \ldots$.)

(f) Write down the expression for $v(x,t)$.

(g) Write down the expression for $u(x,t)$.

Problem 2. Nonlinear equations are very difficult to solve because for them the Principle of Superposition does not hold, and separation of variables does not work. Some nonlinear equations, however, can be simplified by means of a clever transformation. In this problem you will solve the following BVP for a *nonlinear* PDE for the function $u(x,t)$:

$$\begin{aligned} &u_t = u_{xx} + u_x^2 \,, \qquad x \in [0,\pi] \,, \quad t > 0 \,, \\ &u(0,t) = 0 \,, \\ &u(\pi,t) = 0 \,, \\ &u(x,0) = \ln\left(1 + \tfrac{1}{2}\sin 3x\right) \qquad (\text{“ln” is the natural logarithm}) \,. \end{aligned} \tag{2}$$

(a) Perform the *Hopf-Cole transformation*, i.e., define the new function $w(x,t)$ as

$$w(x,t) = \mathrm{e}^{u(x,t)} \,. \tag{3}$$

Use the Chain Rule and the Product Rule to obtain expressions for w_t, w_x, and w_{xx}, and show that $w(x,t)$ satisfies the ordinary heat equation,

$$w_t(x,t) = w_{xx}(x,t) \,, \qquad x \in [0,\pi] \,, \quad t > 0 \,. \tag{4}$$

(b) Use (3) to derive the BCs and the IC for the new unknown function $w(x,t)$ from the BVP (2) for $u(x,t)$.

(c) Use the suggested on page 348 of the book method of “homogenizing” the BCs of a BVP for $w(x,t)$. It is quite easy to see that the “equilibrium” temperature distribution will be $w_\infty(x) = 1$, so that you have to look for solution of the BVP for $w(x,t)$ in the form

$$w(x,t) = 1 + v(x,t) \,.$$

What BVP does the function $v(x,t)$ satisfy?

(d) Use that the solution of the heat equation $v_t = v_{xx}$ for $x \in [0,\pi]$ with homogeneous Dirichlet BCs has the form

$$v(x,t) = \sum_{n=1}^{\infty} A_n \mathrm{e}^{-n^2 t} \sin nx \,.$$

to solve to BVP for $v(x,t)$ derived in part (c).

Hint: You will obtain that the expression for $v(x,t)$ will consist of a single term.

(e) Write down the function $w(x,t)$ and the solution $u(x,t)$ of the original BVP (2).

(f) Let the function $s(x,t)$ be defined as $s(x,t) = u_x(x,t)$. Differentiate the PDE in the BVP (2) with respect to x and rewrite the result as an BVP for the function $s(x,t)$. The equation you derived is called the *Burgers' equation*; it is a popular simple model for phenomena in gas dynamics and traffic flow.

Problem 3. In this problem you will study the waves in a string of length L that is hanging vertically down from a fixed point on the ceiling. Choose the origin of the coordinate system to be at the point where the lower end of the string is when the string is hanging at rest, i.e., the origin is at a distance L under the point where the string is suspended. Let the positive direction of the x-axis be vertically upward. We assume that the string moves in the (x, y)-plane, and its position at time t is described by the equation $y = u(x, t)$.

One can show that the motion of the string is governed by the PDE

$$u_{tt}(x,t) = g\,\frac{\partial}{\partial x}\big[x\,u_x(x,t)\big]\ , \qquad x \in [0, L]\ , \quad t > 0\ , \tag{5}$$

where $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the free-fall acceleration. Physical reasoning shows that the free end of the string (at $x = 0$) satisfies a homogeneous Neumann BC,

$$u_x(0,t) = 0\ . \tag{6}$$

(a) What BC does the string satisfy at $x = L$ (i.e., at the suspension point)?

(b) We will change the variable x to a new variable, s, by

$$s = \widetilde{s}(x) := 2\sqrt{\frac{x}{g}}\ , \tag{7}$$

or, equivalently,

$$x = \widetilde{x}(s) = \frac{g}{4}\,s^2\ .$$

Here $\widetilde{s}$ and $\widetilde{x}$ are functions given by the explicit expressions above.

We define a new function, $v(s,t)$, by

$$v(s,t) := u(x,t)\big|_{x=\widetilde{x}(s)} = u(\widetilde{x}(s), t)\ , \tag{8}$$

or, equivalently, by

$$u(x,t) = v(s,t)\big|_{s=\widetilde{s}(x)} = v(\widetilde{s}(x), t)\ .$$

To write the PDE (5) in terms of the function $v(s,t)$, we compute:

$$u_t(x,t) = \frac{\partial}{\partial t}\,v(\widetilde{s}(x),t) = v_t(\widetilde{s}(x),t)\ ,$$

$$u_x(x,t) = \frac{\partial}{\partial x}\,v(\widetilde{s}(x),t) = v_s(\widetilde{s}(x),t)\,\frac{\mathrm{d}\widetilde{s}}{\mathrm{d}x} = \frac{1}{\sqrt{gx}}\,v_s(\widetilde{s}(x),t)\ ,$$

$$\frac{\partial}{\partial x}\big[x\,u_x(x,t)\big] = \frac{\partial}{\partial x}\left[x\,\frac{1}{\sqrt{gx}}\,v_s(\widetilde{s}(x),t)\right] = \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x}\left[\sqrt{x}\,v_s(\widetilde{s}(x),t)\right]$$

$$= \frac{1}{\sqrt{g}}\left[\frac{1}{2\sqrt{x}}\,v_s(\widetilde{s}(x),t) + \sqrt{x}\,v_{ss}(\widetilde{s}(x),t)\,\frac{\mathrm{d}\widetilde{s}}{\mathrm{d}x}\right]$$

$$= \frac{1}{2\sqrt{gx}}\,v_s(\widetilde{s}(x),t) + \frac{1}{g}\,v_{ss}(\widetilde{s}(x),t)\ .$$

Plugging all these expressions in the PDE (5), we obtain

$$v_{tt}(s,t) = \frac{1}{s}\, v_s(x,t) + v_{ss}(s,t) \ . \tag{9}$$

Write down the interval where the new variable, s, is taking values, as well as the BCs at the two ends of the string (coming from what you found in part (a) and the condition (6)).

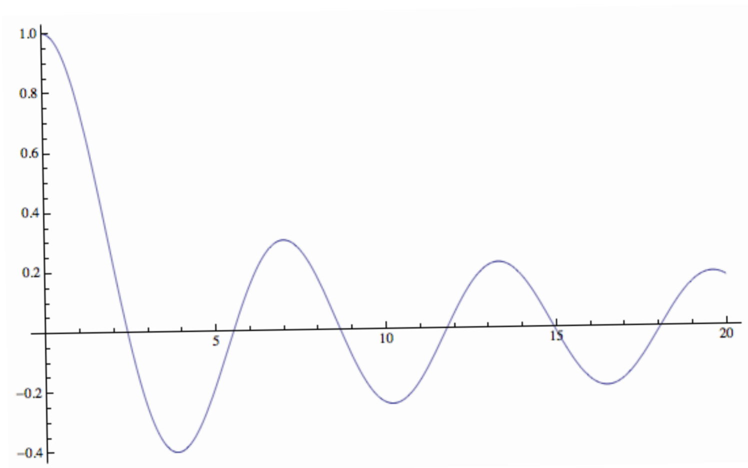
(c) Separate variables in the PDE (9) as usual: set $v(s,t) = S(s)\,T(t)$. The sign of the separation of variables constant must be such that the function $T(t)$ must be oscillatory (i.e., $T(t)$ must have cosines and sines). What ODE does the function $S(s)$ satisfy?

(d) Change variables as on pages 306–307 of the book and express the function $S(s)$ in terms of the Bessel functions $J_0(\xi)$ and Neumann functions $Y_0(\xi)$.

(e) Some of the functions obtained in part (d) behave non-physically. Which ones? Write down the expression for $S(s)$ if we want it to behave in a physically reasonable way.

(f) Does the function found in part (e) satisfy the BC at $x = 0$? Discuss briefly.

Hint: The graph of $J_0(\xi)$ for $\xi \in [0, 20]$ is plotted in the figure below.



(g) Impose the BC at $x = L$, and obtain the values that the separation of variables constant can take. Let ξ_{0k} be the kth zero of $J_0(\xi)$, i.e., $J_0(\xi_{0k}) = 0$, ordered so that $\xi_{01} < \xi_{02} < \cdots$. The values of ξ_{0k} are available in MAPLE and Mathematica; here are the approximate values of the first several zeros: $\xi_{01} = 2.40483$, $\xi_{02} = 5.52008$, $\xi_{03} = 8.65373$, $\xi_{04} = 11.7915$, $\xi_{05} = 14.9309$, ... (look at the graph above).

(h) Write the functions $T_n(t)$ and the solution of the BVP for $v(s,t)$. (We have not imposed initial conditions, so that your expression will contain arbitrary constants.)

(i) Write the function $u(x,t)$ giving the position of the suspended string. (Again, it will contain arbitrary constants.)