MATH 4073 **Homework 7** **Due Wed, 11/2/16**

Problem 1. In this problem you will apply two different methods to approximate the first derivative of the function $f(x) = \sin x$ at $x = \frac{\pi}{3}$.

(a) Apply the forward 2-point formula,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{f''(\xi)}{2}\, h \ , \qquad \xi \in [x_0, x_0+h] \ ,$$

with $h = 0.01$ to find the approximate value of $f'\left(\frac{\pi}{3}\right)$. Find the rigorous error bound,

$$\text{Error} \ \leq \ \frac{h}{2} \max_{\xi\in[x_0,x_0+h]} \left|f''(\xi)\right|$$

(see equation (4.1) in the book). Compute the true value of the derivative $f'\left(\frac{\pi}{3}\right)$, find the true absolute error and compare with the rigorous bound.

(b) Apply the 3-point midpoint formula,

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} - \frac{f'''(\xi_1)}{6}\, h^2 \ , \qquad \xi_1 \in [x_0-h, x_0+h] \ ,$$

with $h = 0.01$ to find the approximate value of $f'\left(\frac{\pi}{3}\right)$. Find the rigorous error bound. Compute the true value of the absolute error and compare with the rigorous bound.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function, and suppose that $x_0 \in \mathbb{R}$ and $h > 0$ are some fixed numbers. Derive a formula to approximate $f'(x_0)$ that uses only $f(x_0 - 2h)$, $f(x_0)$, and $f(x_0 + h)$, whose local truncation error is $O(h^2)$, i.e., such that

$$f'(x_0) = [\,\text{your expression approximating } f'(x_0)\,] + O(h^2) \ .$$

Hint: Expand $f(x_0 - 2h)$, $f(x_0)$, and $f(x_0 + h)$ in a Taylor series about x_0, as in

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}\, h^2 + O(h^3) \ ,$$

and from these expressions eliminate the terms containing $f''(x_0)$.

Problem 3. The forward-difference formula can be expressed as

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{f''(x_0)}{2}\, h - \frac{f'''(x_0)}{6}\, h^2 + O(h^3) \ .$$

Use Richardson extrapolation to derive an $O(h^2)$ formula for $f'(x_0)$.

Problem 4. Let $N_1(h)$ be an approximation to M for every $h > 0$ and

$$M = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

for some constants K_1, K_2, K_3, ... Suppose that that you know the values of $N_1(h)$, $N_1(\frac{h}{3})$, and $N_1(\frac{h}{9})$.

(a) Derive a formula for $N_2(h)$ which is based on the values of $N_1(h)$ and $N_1(\frac{h}{3})$ and provides an $O(h^2)$ approximation to M.

(b) Derive a formula for $N_3(h)$ which is based on the values of $N_2(h)$ and $N_2(\frac{h}{3})$ and provides an $O(h^3)$ approximation to M.

(c) Let $f(x)$ be a smooth function. Use the expansion of $f(x_0+h)$ in a Taylor series around x_0 to derive the formula

$$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \frac{f''(x_0)}{2}\,h - \frac{f'''(x_0)}{6}\,h^2 - \frac{f^{(4)}(x_0)}{24}\,h^3 - \cdots .$$

Now let M stand for the exact value of $f'(x_0)$ and $N(h) := \dfrac{f(x_0+h)-f(x_0)}{h}$. Explain briefly why this implies that Richardson extrapolation can be applied to the forward-differentce formula for approximating first derivatives (equation (4.1) in the book).

(d) Apply the formulae obtained in (a) and (b) to compute the $O(h^2)$ and $O(h^3)$ approximations to the derivative of the function $f(x) = \sin x$ at $x_0 = \frac{\pi}{3}$ for $h = 0.09$ by using Richardson's extrapolation applied to the forward-difference formula for $f'(x_0)$. Compute the numerical values of the actual errors, $|N_j(0.09) - f'(\frac{\pi}{3})|$, for $j = 1, 2, 3$.