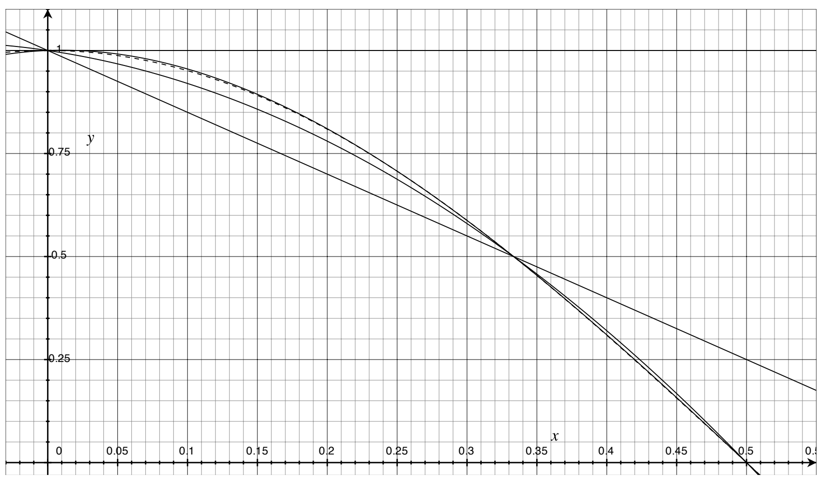
MATH 4073 Homework 7 Due Fri, 10/29/10

Problem 1. The purpose of this problem is to construct and study the Newton's divided difference form of the interpolating polynomial,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j) \ ,$$

to the function $f(x) = \cos(\pi x)$. The points x_i, $i = 0, 1, 2, 3$ used to construct the interpolating polynomial are given in the table below.

The graphs of the function $f(x)$ and the interpolating polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$ are plotted in the figure below (the graph of $f(x)$ is plotted with a dashed line).



(a) Compute the missing entries in the divided differences table below. Write your calculations clearly and leave the coefficients in symbolic form (i.e., do not compute the *numerical values* of things like $12(8\sqrt{2} - 11)$).

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = 0$	$f[x_0] = 1$			
		$f[x_0, x_1] =?$		
$x_1 = \frac{1}{3}$	$f[x_1] = \frac{1}{2}$		$f[x_0, x_1, x_2] =?$	
		$f[x_1, x_2] =?$		$f[x_0, x_1, x_2, x_3] = 12(8\sqrt{2} - 11)$
$x_2 = \frac{1}{2}$	$f[x_2] = 0$		$f[x_1, x_2, x_3] = 12(2\sqrt{2} - 3)$	
		$f[x_2, x_3] = -2\sqrt{2}$		
$x_3 = \frac{1}{4}$	$f[x_3] =?$			

(b) Write down the interpolating polynomial $P_0(x)$ based on the values in the divided differences table above.

(c) Similarly to part (b), write down the interpolating polynomial $P_1(x)$ based on the values in the divided differences table above.

(d) Similarly to part (b), write down the interpolating polynomial $P_2(x)$ based on the values in the divided differences table above. Do *not* expand it – just substitute the coefficients in the Newton's divided difference interpolating polynomial with the corresponding entries from the table.

(e) Similarly to part (b), write down the interpolating polynomial $P_3(x)$ based on the values in the divided differences table above. Do *not* expand the polynomial!

Problem 2. Use Newton's forward-difference formula to construct interpolating polynomials for the function $f(x) = \cos(\pi x)$ based on values of $f(x)$ at equally spaced points (note that the points x_0, x_1, x_2 and x_3 below are equally spaced with $h = \frac{1}{6}$). First write the polynomial in the form

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0) \ ,$$

where $x = x_0 + sh$, i.e., $s = \frac{x - x_0}{h}$.
Fill the missing entries in the forward differences table below, and find the following:

(a) the degree-1 forward difference polynomial $P_1(x)$ based on the points $(x_0, f(x_0)) = (0, 1)$ and $(x_1, f(x_1)) = (\frac{1}{6}, \frac{\sqrt{3}}{2})$;

(b) the degree-2 forward difference polynomial $P_2(x)$ based on the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2)) = (\frac{1}{3}, ?)$;

(c) the degree-3 forward difference polynomial $P_3(x)$ based on the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3)) = (\frac{1}{2}, ?)$.

(d) In Mathematica (or any other computer system), plot the graphs of $\cos(\pi x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.

Use the forward differences table below, some of whose entries are given. Do *not* compute the numerical values (i.e., leave the square roots, as in the table below, and, more importantly, leave the polynomial in Newton's forward-diference form (i.e., do *not* expand it).

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = 0$	$f(x_0) = 1$			
		$\Delta f(x_0) = ?$		
$x_1 = \frac{1}{6}$	$f(x_1) = \frac{\sqrt{3}}{2}$		$\Delta^2 f(x_0) = ?$	
		$\Delta f(x_1) = ?$		$\Delta^3 f(x_0) = \frac{3\sqrt{3}-5}{2}$
$x_2 = \frac{1}{3}$	$f(x_2) = ?$		$\Delta^2 f(x_1) = \frac{\sqrt{3}-2}{2}$	
		$\Delta f(x_2) = -\frac{1}{2}$		
$x_3 = \frac{1}{2}$	$f(x_3) = ?$			

Problem 3. Consider the function $f(x) = \sqrt{x}$. We want to find a cubic polynomial $S(x)$ that interpolates the function $f(x)$ on the interval $x \in [1, 4]$ with clamped boundary conditions at both ends.

(a) Clearly, $S(x)$ must have the same values as $f(x)$ at the points $x_0 = 1$ and $x_1 = 4$. What are the clamped boundary conditions for $S'(1)$ and $S'(4)$?

(b) Write the interpolating polynomial $S(x)$ in the form

$$S(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 \ ,$$

and impose the conditions formulated in part (a) to find the coefficients of $S(x)$. You should obtain that $c = -\frac{1}{12}$ and $d = \frac{1}{108}$.

(c) Apply Theorem 3.13 on page 152 to give a rigorous upper bound on the interpolation error. Compare this numerical value with the actual value of the error, which is about 0.013.

(d) The function $S(x)$ that you found in (a) is good not only to approximate the values of the function $f(x)$, but also to approximate the values of the integrals and some derivatives of $f(x)$. Find the numerical value of $\int_1^3 S(x)\, dx$ and compare it with the exact value, $\int_1^3 f(x)\, dx$; find the absolute and the relative errors.

Problem 4. The polynomials

$$\begin{aligned} S_0(x) &= 4 + 4(x-1) + 13(x-1)^2 - 9(x-1)^3 \ , \quad x \in [1,2] \ , \\ S_1(x) &= a + b(x-2) + c(x-2)^2 + d(x-2)^3 \ , \quad x \in [2,3] \end{aligned}$$

form the *clamped* cubic spline interpolant for some function $f(x)$ that is known to satisfy $f'(1) = -f'(3)$.

(a) Use this information to find the values of the coefficients a, b, c, and d.

(b) Compute $S'(2.5)$.