

**Problem 1.** This problem is about different methods for approximate integration applied to the integral  $\int_0^{\pi/3} \tan x \, dx$ . In this problem “trapezoidal rule” and “midpoint rule” always refer to the “simple” rules, i.e., the ones derived in Section 4.3 of the book (not the “composite” ones discussed in Section 4.4).

- Compute by hand the indefinite integral  $\int \tan x \, dx$ ; show me your solution in detail. Find the exact value of the definite integral  $\int_0^{\pi/3} \tan x \, dx$ .
- Is the function  $\tan x$  increasing on  $[0, \frac{\pi}{3}]$ ? Is it concave up or concave down on  $[0, \frac{\pi}{3}]$ ? Justify your answers.
- Based on your answer in part (b), can you predict whether the trapezoidal rule applied to  $\int_0^{\pi/3} \tan x \, dx$  will give you a smaller or a larger value than the true value of the integral (obtained in part (a))? Explain and draw a sketch to support your claim.
- Apply the trapezoidal rule to the integral  $\int_0^{\pi/3} \tan x \, dx$ . Does your result agree with your prediction in part (c)?
- Based on your answer in part (b), can you predict whether the midpoint rule applied to  $\int_0^{\pi/3} \tan x \, dx$  will give you a smaller or a larger value than the true value of the integral (obtained in part (a))? Explain and draw a sketch to support your claim.
- Apply the midpoint rule to the integral  $\int_0^{\pi/3} \tan x \, dx$ . Does your result agree with your prediction in part (e)?

**Problem 2.** This problem is about the “simple” (see Section 4.3) and the “composite” (see Section 4.4) Simpson’s rules for approximate integration applied to the definite integral  $\int_0^{\pi/3} \tan x \, dx$ . In some parts of this problem the calculations are long, so Mathematica will be helpful.

- Find the theoretical upper bound on the (absolute) error in applying the “simple” Simpson’s rule to the integral  $\int_0^{\pi/3} \tan x \, dx$ ; you may use that

$$\frac{d^4}{dx^4} \tan x = \frac{4(5 - \cos 2x) \tan x}{\cos^4 x} ;$$

one can easily see that this function is increasing on  $[0, \frac{\pi}{3}]$  by noticing that  $\cos$  is decreasing and  $\tan$  is increasing on this interval. You can use that

$$\left. \frac{d^4}{dx^4} \tan x \right|_{x=\frac{\pi}{3}} = 352\sqrt{3} .$$

- (b) Compute the approximate value of the definite integral  $\int_0^{\pi/3} \tan x \, dx$  by using the “simple” Simpson’s rule.

*Hint:* The stepsize is  $h = \frac{\pi}{6}$ .

- (c) Find the true value of the (absolute) error of the calculation from part (b) (you have found the true value of the definite integral in Problem 1(a)). Compare it with the rigorous bound obtained in part (a).

- (d) Now you will apply the composite Simpson’s rule to compute the approximate value of  $\int_0^{\pi/3} \tan x \, dx$  by dividing the interval  $[0, \frac{\pi}{3}]$  into  $n = 4$  equal-size subintervals. What is the value of  $h$ ? Use that the composite Simpson’s rule for  $n$  equal subintervals is (see Section 4.4)

$$\int_a^b f(x) \, dx = \frac{h}{3} \left[ f(x_0) + 2 \sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right] - \frac{b-a}{180} h^4 f^{(4)}(\mu) ,$$

where  $\mu$  is some (unknown) value in  $[0, \frac{\pi}{3}]$ , to give a rigorous upper bound on the (absolute) error in the computation of the integral by applying the composite Simpson’s rule with  $n = 4$ .

*Hint:* In this case  $h = \frac{\pi}{12}$ . You can use that  $\tan \frac{\pi}{12} = 2 - \sqrt{3}$ .

- (e) Compute the approximate value of the integral  $\int_0^{\pi/3} \tan x \, dx$  by using the formula from part (d).
- (f) What is the true value of the (absolute) error in the calculation from part (e)? Compare it with the rigorous bound from part (d).
- (g) Finally, recall that in the process of deriving the composite Simpson’s rule, we had that the error was equal to

$$-\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) ,$$

where  $\xi_j \in [x_{2j-2}, x_{2j}]$ ; we used this expression and the Extreme and Intermediate Value Theorems to derive the bound given in part (d). In fact, the more complicated expression given here gives a better bound (but is not used very often because it is more complicated). Explain why this rigorous upper bound is tighter, i.e., smaller (think about the derivation of the simplified bound in class). Then apply the expression written here to obtain a better rigorous upper bound on the error in your computation from part (e). Compare it with the rigorous upper bound obtained in part (d). Here are some useful values:

$$\frac{d^4}{dx^4} \tan x \Big|_{x=\frac{\pi}{3}} = 352\sqrt{3} , \quad \frac{d^4}{dx^4} \tan x \Big|_{x=\frac{\pi}{6}} = \frac{32}{\sqrt{3}} .$$

**Problem 3.** In this problem you will study empirically how the true errors in the composite trapezoidal and Simpson's rules depend on the stepsize  $h = \frac{b-a}{n}$  in computing the definite integral  $\int_a^b f(x) dx$ . As an example, you will use  $\int_0^{\pi/3} \tan x dx$  (whose exact value was found in Problem 1(a)).

Let  $E_n$  be the absolute error in the calculation, i.e., the absolute value of the difference between the exact value of the integral and the approximate value computed by dividing the interval  $[a, b]$  into  $n$  subintervals of equal size.

- (a) If  $E_n \approx C h^\alpha$  for some constant  $C > 0$  and some value of  $\alpha$ , then what is the (approximate) dependence of  $E_n$  on  $n$ ? If you plot  $\ln E_n$  versus  $\ln n$ , then the points  $(\ln n, \ln E_n)$  will be approximately on a straight line. What is the slope of this straight line? Prove your claim.

- (b) Show that if  $E_n \approx C h^\alpha$ , then

$$\ln \frac{E_n}{E_{2n}} \approx \alpha \ln 2 .$$

- (c) The Matlab code `comp_trap.m` (on the class web-site) implements the composite trapezoidal rule. Use it to compute the approximate values of  $\int_0^{\pi/3} \tan x dx$  with  $n = 16, 32, 64, 128, 256,$  and  $512$ , and the corresponding absolute errors.
- (d) Use your data from part (c) and the formula obtained in part (b) to find the value of  $\alpha$  for the composite trapezoidal rule. What value of  $\alpha$  did you expect, and why? Compare it with the empirically obtained value. Discuss briefly.
- (e) The Matlab code `comp_simp.m` (on the class web-site) implements the composite Simpson's rule. Use it to compute the approximate values of  $\int_0^{\pi/3} \tan x dx$  with  $n = 16, 32, 64, 128, 256,$  and  $512$ , and the corresponding absolute errors.
- (f) Use your data from part (e) and the formula obtained in part (b) to find the value of  $\alpha$  for the composite Simpson's rule. What value of  $\alpha$  did you expect, and why? Compare it with the empirically obtained value. Discuss briefly.