MATH 3423 Homework 7 Due Tue, 10/22/19

Problem 1. Show that, if f is analytic on and inside a simple closed contour C, and if the point a is not on the contour C, then

$$\oint_C \frac{f'(z)}{z-a}\,\mathrm{d}z = \oint_C \frac{f(z)}{(z-a)^2}\,\mathrm{d}z\ .$$

Hint: Apply the generalization of the Cauchy Integral Formula.

Remark: Consider separately the case when a is outside of the contour C, a and when a is inside C. Also, recall that the derivative of an analytic function is also analytic.

Problem 2. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,\mathrm{d}x}{(1+x^2)^2} = \frac{\pi}{2}$ by using the contour in Figure 19.5 on p. 931.

Hint: You may use without proving it that the integral over the semi-circle C_R given by $\{z \in \mathbb{C} : |z| = R, \operatorname{Im} z > 0\}$ tends to zero as $R \to \infty$: indeed, for large R, $(1+z^2)^2 \approx z^4$, so

$$\left|\int_{C_R} \frac{z^2\,\mathrm{d}z}{(1+z^2)^2}\right| \approx \left|\int_{C_R} \frac{z^2\,\mathrm{d}z}{z^4}\right| = \left|\int_{C_R} \frac{\mathrm{d}z}{z^2}\right| \le \max_{z\in C_R}\left|\frac{1}{z^2}\right| (\text{length of } C_R) = \frac{\pi R}{R^2} \to 0 \quad \text{as } R \to \infty\,.$$

Problem 3. The curves in $\mathbb{C}$ defined by the equations

$$Z_1(t) = 1 + 5t + it^3\ , \qquad Z_2(t) = \mathrm{e}^t + i\sin\big(t\sqrt{3}\big) \tag{1}$$

(where $t \in \mathbb{R}$) intersect for $t = 0$ at the point 1 (i.e., $Z_1(0) = Z_2(0) = 1$).

(a) Find $Z_1'(0)$ and $Z_2'(0)$.

(b) Find the angle between the tangents to $Z_1(t)$ and $Z_2(t)$ at their intersection for $t = 0$. Draw a picture to make your argument more clear.

Hint: Write the complex numbers $Z_1'(0)$ and $Z_2'(0)$ in polar coordinates. How can you find the angle between the vectors $Z_1'(0)$ and $Z_2'(0)$ from the arguments of the complex numbers $Z_1'(0)$ and $Z_2'(0)$?

(c) Now think of the two curves in $\mathbb{C}$ defined by (1) as curves in $\mathbb{R}^2$, namely,

$$\mathbf{r}_1(t) = (1+5t)\,\mathbf{i} + t^3\,\mathbf{j}\ , \quad \mathbf{r}_2(t) = \mathrm{e}^t\,\mathbf{i} + \sin\big(t\sqrt{3}\big)\,\mathbf{j}\ ,$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in positive x- and y-directions, resp. Find $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$.

(d) Find $\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0)$ and use this to compute the angle between the curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the point of their intersection for $t = 0$.

Hint: For any vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors.

Problem 4. Consider the transformation

$$f : \left\{ z \in \mathbb{C} : \operatorname{Im} z \in \left(0, \tfrac{\pi}{2}\right) \right\} \to \left\{ w \in \mathbb{C} : \operatorname{Arg} w \in \left(0, \tfrac{\pi}{2}\right) \right\} \ ,$$

defined by $w = f(z) = \mathrm{e}^z$, which maps the infinite horizontal strip

$$\left\{ z \in \mathbb{C} : \operatorname{Im} z \in \left(0, \tfrac{\pi}{2}\right) \right\}$$

to the first quadrant,

$$\left\{ w \in \mathbb{C} : \operatorname{Arg} w \in \left(0, \tfrac{\pi}{2}\right) \right\} \ .$$

The transformation is represented graphically in Figure 1.

(a) Show that the points A and B shown in the figure go to the points A' and B', respectively.

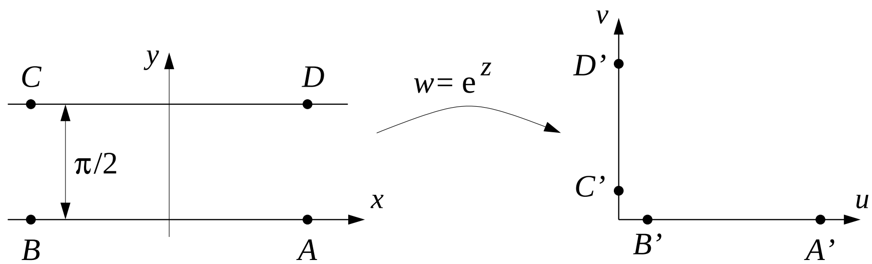

Figure 1: The transformation $w = f(z) = \mathrm{e}^z$.

(b) Draw the straight line $\{y = \frac{\pi}{4}\}$ in the left part of the figure above. Find the image of this line under the map f and draw it in the right part of the figure above.

(c) Draw the straight line $\{x = 1\}$ in the left part of the figure above. Find the image of this line under the map f and draw it in the right part of the figure above.

(d) In parts (b) and (c) you found the images of the straight lines $\{y = \frac{\pi}{4}\}$ and $\{x = 1\}$ under the map f. What is the angle between the *images* of these two lines at the point where they intersect? You can either compute it, or figure it out using some facts that we have learned. Computing it is a relatively long calculations; on the other hand, using the general mathematical results will answer the question with almost no effort. Please write specifically which results you are using in your reasoning.

Problem 5. This problem is related to Problem 4 above. Let

$$\psi : \left\{ w \in \mathbb{C} : \operatorname{Arg} w \in \left(0, \tfrac{\pi}{2}\right) \right\} \to \mathbb{R}$$

be a real-valued function defined on the first quadrant in $\mathbb{C}$ and given by

$$\psi(u, v) = (u^2 + v^2)^5 + 3 \arctan \frac{v}{u} \ . \tag{2}$$

Here we identify a complex number $w = u + iv \in \mathbb{C}$ (where $u \in \mathbb{R}$ and $v \in \mathbb{R}$) with a point $(u, v) \in \mathbb{R}^2$. As in Problem 3, define the transformation

$$f : \left\{z \in \mathbb{C} : \operatorname{Im} z \in \left(0, \tfrac{\pi}{2}\right)\right\} \to \left\{w \in \mathbb{C} : \operatorname{Arg} w \in \left(0, \tfrac{\pi}{2}\right)\right\} ,$$

given by $w = f(z) = \mathrm{e}^z$. Write $z = x + iy$ and $w = u + iv$, so that

$$u + iv = w = f(z) = \mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x \mathrm{e}^{iy} = \mathrm{e}^x(\cos y + i \sin y) . \tag{3}$$

(a) Consider (3) as defining a change of variables

$$u = U(x, y) , \quad v = V(x, y) .$$

Write the functions $U(x, y)$ and $V(x, y)$ explicitly.

(b) Change variables in the function ψ defined in (2) from (u, v) to (x, y), where the change was obtained in part (a). In other words, define the function

$$\phi : \left\{z \in \mathbb{C} : \operatorname{Im} z \in \left(0, \tfrac{\pi}{2}\right)\right\} \to \mathbb{R}$$

by

$$\phi(x, y) := \psi\left(U(x, y), V(x, y)\right) .$$

Find the explicit expression for $\phi(x, y)$.

Problem 6. In this problem you will solve a boundary value problem for the Laplace's equation (describing the steady-state temperature distribution) in the complicated domain

$$D := \left\{z \in \mathbb{C} : \operatorname{Re} z > 0, \left|z - \tfrac{1}{2}\right| > \tfrac{1}{2}\right\} , \tag{4}$$

which is drawn in the left part of Figure 7 on page 964 of the book. The boundary ∂D of the domain D consists of the imaginary axis, i.e., the straight line $C_1 = \{\operatorname{Re} z = 0\}$, and the circle $C_2 = \left\{\left|z - \tfrac{1}{2}\right| = \tfrac{1}{2}\right\}$. The boundary value problem you will solve below is

$$\begin{aligned} &\Delta\phi(x, y) = 0 , \quad (x, y) \in D , \\ &\phi|_{C_1} = \alpha , \\ &\phi|_{C_2} = \beta , \end{aligned} \tag{5}$$

where α and β are real constants; here $\phi|_{C_j}$ stands for the restriction of the function ϕ to the curve C_j, $j = 1, 2$.

(a) Consider the transformation $w = \frac{1}{z}$, which maps the domain D defined in (4) to the infinite vertical strip

$$\widetilde{D} = \{z \in \mathbb{C} : \operatorname{Re} z \in (0, 1)\} \tag{6}$$

(see Figure 7 on page 964 of the book). Consider $w = \frac{1}{z}$ as defining a change of variables $u = U(x, y)$, $v = V(x, y)$. Write down the functions $U(x, y)$ and $V(x, y)$.

Hint: The expression for $V(x, y)$ is $V(x, y) = -\dfrac{y}{x^2 + y^2}$ (but I want to see your detailed calculations).

(b) The inverse transformation, $z = \frac{1}{w}$, maps the infinite vertical strip $\widetilde{D}$ (6) back to the domain D (4). Write down the explicit expressions for the functions $X(u,v)$ and $Y(u,v)$ in the inverse transformation.

Hint: This can be done VERY easily! Do the "direct", $w = \frac{1}{z}$, and the "inverse", $z = \frac{1}{w}$, transformations look similar? Can you use this fact in order to reuse your calculations in part (a) to answer the question in part (b).

(c) As you know, an analytic function, like the one defining the transformation $w = \frac{1}{z}$ (for $z \neq 0$), defines a conformal transformation which, in turn, transforms the Laplace's equation to Laplace's equation. Therefore, the function $\psi : \widetilde{D} \to \mathbb{C}$ defined by

$$\psi(u,v) := \phi(X(u,v), Y(u,v))$$

satisfies the Laplace's equation in the infinite vertical strip $\widetilde{D}$:

$$\Delta\psi(u,v) = \psi_{uu}(u,v) + \psi_{vv}(u,v) = 0 \ , \quad (u,v) \in \widetilde{D}$$

(you do *not* have to prove this here!). The straight line C_1 goes into the vertical line $\widetilde{C}_1 = \{\operatorname{Re} w = 0\}$, and the circle C_2 goes into the vertical line $\widetilde{C}_2 = \{\operatorname{Re} w = 1\}$, as you can see from Figure 7 on page 964. Write explicitly the boundary value problem for the function ψ, i.e.,

$$\begin{aligned} &\Delta\psi(u,v) = ? \ , \quad (u,v) \in \widetilde{D} \ , \\ &\psi|_{\widetilde{C}_1} = ? \ , \\ &\psi|_{\widetilde{C}_2} = ? \ . \end{aligned}$$

(d) What is the solution of the boundary value problem written in part (c)? Guess what it is, and that will be the solution! (Thanks to the theorem guaranteeing the uniqueness of solution of this problem.)

Hint: The solution $\psi(u,v)$ in fact depends only on u, not on v.

(e) Now transform back to the (x,y) variables to obtain the solution $\phi(x,y)$ of the initial boundary value problem (5).

Problem 7. Use the fact that the Fourier transform of the function $f(x) = \mathrm{e}^{-|x|}$ is

$$\widehat{F}(k) = \sqrt{\frac{2}{\pi}} \, \frac{1}{1+k^2}$$

and Parseval's Theorem to show that

$$\int_0^\infty \frac{\mathrm{d}y}{(1+y^2)^2} = \frac{\pi}{4} \ .$$

Problem 8. Let α be a positive real number, and the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \mathrm{e}^{-\alpha|x|} \ ,$$

Use without proof that

$$\widehat{F}(k) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{k^2 + \alpha^2} \ ,$$

as well as some property of Fourier transform (specify which property!) to show that the Fourier transform of the function

$$g(x) = x\mathrm{e}^{-\alpha|x|}$$

is

$$\widehat{G}(k) = -\sqrt{\frac{2}{\pi}} \frac{2\alpha i k}{(k^2 + \alpha^2)^2} \ .$$

Problem 9. Consider the following initial value problem for the function $u(x,t)$:

$$\frac{1}{v}\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \ , \qquad x \in \mathbb{R} \ , \quad t > 0 \ ,$$

$$u(x,0) = f(x) \ ,$$

where v is a positive real constant.

(a) Let the Fourier transform with respect to x of $u(x,t)$ be

$$\widehat{U}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, \mathrm{e}^{-ikx}\, \mathrm{d}x$$

and its inverse be

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{U}(k,t)\, \mathrm{e}^{ikx}\, \mathrm{d}k \ .$$

Formulate an initial value problem for the function $\widehat{U}(k,t)$ of the form

$$\frac{\partial \widehat{U}}{\partial t}(k,t) = \ ? \ , \qquad k \in \mathbb{R} \ , \quad t > 0 \ ,$$

$$\widehat{U}(k,0) = \ ? \ .$$

(b) Find the solution $\widehat{U}(k,t)$ of the initial value problem formulated in part (a).

(c) Use some of the properties of the Fourier transform (state explicitly which property you have used!) to show that the solution of the initial value problem is

$$u(x,t) = f(x + vt) \ .$$