MATH 3423 **Homework 7** **Due Mon, 10/31/16**

Problem 1. Consider the following vectors in $\mathbb{R}^3$: $\mathbf{u} := \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$, $\mathbf{v} := \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$, and $\mathbf{w} := \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$. Directly from the definition, find out whether they are linearly dependent. Please write all your calculations in detail.

Problem 2. Let V be a two-dimensional linear space, and $\mathsf{A} : V \to V$ be a linear operator acting on V. Let the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ form a basis in V, and the linear operator A acts on these two vectors as follows: $\mathsf{A}\mathbf{f}_1 = 2\mathbf{f}_1$, $\mathsf{A}\mathbf{f}_2 = 4\mathbf{f}_1 + \mathbf{f}_2$.

(a) Write down the matrix $\underline{\underline{A}} = (a_{ij})$ of the operator A in the basis $\mathbf{f}_1$, $\mathbf{f}_2$; recall that the entries of $\underline{\underline{A}}$ are defined by

$$\mathsf{A}\mathbf{f}_j =: \sum_{i=1}^{n} a_{ij}\, \mathbf{f}_i \ .$$

(b) If $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$, find $\mathsf{A}\mathbf{u}$ (expressed as a linear combination of the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$).

(c) Show that the vectors $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$ and $\mathbf{v} = 2\mathbf{f}_1 + \mathbf{f}_2$ are linearly independent, i.e., that if $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ (where $\alpha, \beta \in \mathbb{R}$), then $\alpha = 0$ and $\beta = 0$.

Problem 3. Suppose that the linear operator $\mathsf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ transforms $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ into $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ into $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ into $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Find the matrix $\underline{\underline{A}}$ that corresponds to the operator A.

Hint: You may use that, if $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the standard basis in $\mathbb{R}^2$, then $\mathbf{u}_1 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathsf{A}\mathbf{u}_1 = \mathbf{v}_1 = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$, which, by using the fact that A is a *linear* operator, is the same as $\mathsf{A}\mathbf{e}_1 + \mathsf{A}\mathbf{e}_2 = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$; similarly, you will obtain two more equations: $\mathsf{A}\mathbf{e}_2 + \mathsf{A}\mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathsf{A}\mathbf{e}_1 + \mathsf{A}\mathbf{e}_2 =$ (something), from which you can express $\mathsf{A}\mathbf{e}_j$ $(j = 1, 2, 3)$ in terms of the vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$.

Problem 4. Consider the linear space of polynomials of degree no greater than 3. Let us choose $\mathbf{f}_0 = 1$, $\mathbf{f}_1 = x$, $\mathbf{f}_2 = x^2$, $\mathbf{f}_3 = x^3$ to be a basis in this vector space, so that each polynomial $P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$ can be written as a vector $\mathbf{p}$ in this vector space as

$$\mathbf{p} = \sum_{i=0}^{3} p_i\, \mathbf{f}_i \ .$$

(a) Let D be the differentiation operator. Find the matrix $\underline{\underline{D}}$ of D in the basis $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$.

(b) Find the matrix of the operator D^2 (where $\mathsf{D}^2 := \mathsf{DD}$). You may use the fact that the matrix of a composition of the operators A and B is equal to the product of the matrices of A and B.

(c) Find the matrices of D^k for $k = 3, 4, \ldots$. Do you get the zero operator O for some value of k? An operator A for which it happens that $\mathsf{A}^k = \mathsf{O}$ for some k is called a *nilpotent* operator.

Problem 5. Determine the geometric meaning of the operators A, B, and C acting on $\mathbb{R}^2$, if they are represented by the following matrices:

$$\underline{\underline{A}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \underline{\underline{C}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

Hint: Take an arbitrary vector in $\mathbb{R}^2$, say $\mathbf{u} = \binom{2}{1}$, draw $\mathbf{u}$ and at the products $\underline{\underline{A}}\mathbf{u}$, $\underline{\underline{B}}\mathbf{u}$, and $\underline{\underline{C}}\mathbf{u}$ in $\mathbb{R}^2$, and the geometric meaning of the corresponding operators will be transparent.

Problem 6. As we mentioned in class, one can define different norms in the same linear space. In this problem you will study different norms in $\mathbb{R}^2$. Let $\mathbf{u} = \binom{u_1}{u_2} = u_1\mathbf{i} + u_2\mathbf{j} \in \mathbb{R}^2$.

(a) Define the norm $\|\mathbf{u}\|_2$ by $\|\mathbf{u}\|_2 := \sqrt{u_1^2 + u_2^2}$. Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \, : \, \|\mathbf{v}\|_2 \le 1\}$.

(b) Define the norm $\|\mathbf{u}\|_1$ by $\|\mathbf{u}\|_1 := |u_1| + |u_2|$. Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \, : \, \|\mathbf{v}\|_1 \le 1\}$.

(c) Define the norm $\|\mathbf{u}\|_\infty$ by $\|\mathbf{u}\|_\infty := \max\{|u_1|, |u_2|\}$ .. Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \, : \, \|\mathbf{v}\|_\infty \le 1\}$.

(d) Two norms $\| \ \|$ and $\| \ \|'$ on the same linear space are said to be *equivalent* if there exist positive constants C_1 and C_2 such that $C_1 \|\mathbf{u}\| \le \|\mathbf{u}\|' \le C_2 \|\mathbf{u}\|$ for any vector $\mathbf{u} \in V$. Here we will prove that the norms $\| \ \|_2$ and $\| \ \|_\infty$ on $\mathbb{R}^2$ are equivalent:

$$\|\mathbf{u}\|_\infty = \max\{|u_1|, |u_2|\} \le \sqrt{|u_1|^2 + |u_2|^2} = \|\mathbf{u}\|_2 \ ,$$

and

$$\|\mathbf{u}\|_2 = \sqrt{|u_1|^2 + |u_2|^2} \le \sqrt{2 \, \max\{|u_1|^2, |u_2|^2\}} = \sqrt{2} \, \max\{|u_1|, |u_2|\} = \sqrt{2} \, \|\mathbf{u}\|_\infty \ .$$

The inequalities $\|\mathbf{u}\|_\infty \le \|\mathbf{u}\|_2 \le \sqrt{2} \, \|\mathbf{u}\|_\infty$, mean that the norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent (the values of the constants are $C_1 = 1$ and $C_2 = \sqrt{2}$).

Show that the norms $\|\mathbf{u}\|_1$ and $\|\mathbf{u}\|_\infty$ are equivalent (you have to find the corresponding constants $\widetilde{C}_1$ and $\widetilde{C}_2$ such that $\widetilde{C}_1 \, \|\mathbf{u}\|_1 \le \|\mathbf{u}\|_\infty \le \widetilde{C}_2 \, \|\mathbf{u}\|_1$).

(e) Use the fact that the norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent and the fact (proved in part (d)) that the norms $\| \ \|_1$ and $\| \ \|_\infty$ are equivalent to prove that the norms $\|\mathbf{u}\|_1$ and $\|\mathbf{u}\|_2$ are equivalent. In other words, you have to find constants C_1' and C_2' such that $C_1' \|\mathbf{u}\|_1 \le \|\mathbf{u}\|_2 \le C_2' \, \|\mathbf{u}\|_1$. This won't require any additional calculations – simply express the constants C_1' and C_2' in terms of C_1, C_2, $\widetilde{C}_1$, and $\widetilde{C}_2$.