

**Problem 1.** Consider the following three vectors in  $\mathbb{R}^3$ :

$$\mathbf{u} := \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}, \quad \mathbf{v} := \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}, \quad \mathbf{w} := \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}.$$

Directly from the definition, find out whether they are linearly dependent. Please write all your calculations in detail

**Problem 2.** Let  $V$  be a two-dimensional linear space, and  $A : V \rightarrow V$  be a linear operator acting on  $V$ . Let the vectors  $\mathbf{f}_1$  and  $\mathbf{f}_2$  form a basis in  $V$ , and the linear operator  $A$  acts on these two vectors as follows:  $A\mathbf{f}_1 = 2\mathbf{f}_1$ ,  $A\mathbf{f}_2 = 4\mathbf{f}_1 + \mathbf{f}_2$ .

- Write down the matrix of the operator  $A$  in the basis  $\mathbf{f}_1, \mathbf{f}_2$ .
- If  $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$ , find  $A\mathbf{u}$  (expressed as a linear combination of the vectors  $\mathbf{f}_1$  and  $\mathbf{f}_2$ ).
- Show that the vectors  $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$  and  $\mathbf{v} = 2\mathbf{f}_1 + \mathbf{f}_2$  are linearly independent, i.e., that  $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ , (where  $\alpha, \beta \in \mathbb{R}$ ), then  $\alpha = 0$  and  $\beta = 0$ .

**Problem 3.** Let  $V$  be a linear space without any additional structure on it (i.e., there is no norm, inner product, or the concept of orthogonality). Consider the linear operator  $A : V \rightarrow V$  acting on  $V$ . Let  $\mathbf{f}_1, \dots, \mathbf{f}_n$  be a basis in  $V$ . One can define the matrix elements of  $A$  in this basis as follows:

$$A\mathbf{f}_j =: \sum_{i=1}^n a_{ij} \mathbf{f}_i,$$

so that the matrix of  $A$  in this basis is  $\underline{\underline{A}} = (a_{ij})$  (see equation (7) on page 457).

- Let  $\mathbf{u} = \sum_{j=1}^n u_j \mathbf{f}_j$ . Find the components of  $A\mathbf{u}$  in the basis  $\mathbf{f}_1, \dots, \mathbf{f}_n$ ; in other words, if  $\mathbf{v} = A\mathbf{u}$ , find the numbers  $v_i$  such that  $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{f}_i$ .
- Let  $B$  be another linear operator with matrix elements  $(b_{ij})$  in the basis  $\mathbf{f}_1, \dots, \mathbf{f}_n$ . Find the matrix elements of the operator  $AB$  in the same basis. Here  $AB$  stands the composition of the operators in the sense that  $(AB)\mathbf{u} := A(B\mathbf{u})$ .
- Let  $C : V \rightarrow V$  be an *invertible* linear operator, i.e., a linear operator that has an inverse  $C^{-1}$ , so that  $C(C^{-1}) = (C^{-1})C = I$  (where  $I$  is the identity operator:  $I\mathbf{u} = \mathbf{u}$  for any  $\mathbf{u} \in V$ ). Let  $C$  and  $C^{-1}$  have matrices  $\underline{\underline{C}} = (c_{ij})$  and  $\underline{\underline{D}} = (d_{ij})$ , in the basis  $\mathbf{f}_1, \dots, \mathbf{f}_n$ :

$$C\mathbf{f}_j = \sum_{i=1}^n c_{ij} \mathbf{f}_i, \quad C^{-1}\mathbf{f}_j = \sum_{i=1}^n d_{ij} \mathbf{f}_i.$$

Show that the matrix  $\underline{\underline{D}}$  corresponding to  $C^{-1}$  is equal to the inverse of the matrix  $\underline{\underline{C}}$  corresponding to  $C$ , i.e.,  $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$ .

- (d) Let us use the operators  $C$  and  $C^{-1}$  to define a new basis in  $V$ . Namely, let the new basis  $\tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_n$  be defined by

$$\mathbf{f}_j = C \tilde{\mathbf{f}}_j = \sum_{i=1}^n c_{ij} \tilde{\mathbf{f}}_i ;$$

clearly, this implies that

$$\tilde{\mathbf{f}}_j = C^{-1} \mathbf{f}_j = \sum_{i=1}^n d_{ij} \mathbf{f}_i .$$

Prove that if  $\tilde{u}_j$  are the components of  $\mathbf{u}$  in the new basis, i.e.,  $\mathbf{u} = \sum_{j=1}^n \tilde{u}_j \tilde{\mathbf{f}}_j$ , then  $\tilde{u}_i = \sum_{j=1}^n c_{ij} u_j$ .

- (e) Now we want to find the matrix of the operator  $A$  in the new basis  $\tilde{\mathbf{f}}_1, \dots, \tilde{\mathbf{f}}_n$ . Define the matrix elements  $\tilde{a}_{ij}$  of  $A$  in the new basis by

$$A \tilde{\mathbf{f}}_j =: \sum_{i=1}^n \tilde{a}_{ij} \tilde{\mathbf{f}}_i ,$$

and let  $\underline{\underline{\tilde{A}}} = (\tilde{a}_{ij})$ . Show that  $\underline{\underline{\tilde{A}}} = \underline{\underline{C}} \underline{\underline{A}} \underline{\underline{C}}^{-1}$ .

**Problem 4.** Consider the linear space of polynomials of degree no greater than 3. Let us choose  $\mathbf{f}_0 = 1$ ,  $\mathbf{f}_1 = x$ ,  $\mathbf{f}_2 = x^2$ ,  $\mathbf{f}_3 = x^3$  to be a basis in this vector space, so that each polynomial  $P(x) = p_0 + p_1x + p_2x^2 + p_3x^3$  can be written as a vector  $\mathbf{p}$  in this vector space as

$$\mathbf{p} = \sum_{i=0}^3 p_i \mathbf{f}_i .$$

- (a) Let  $D$  be the differentiation operator. Find the matrix  $\underline{\underline{D}}$  of  $D$  in the basis  $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ .
- (b) Find the matrix of the operator  $D^2$  (where  $D^2 := DD$ ). You may use the fact that the matrix of a composition of the operators  $A$  and  $B$  is equal to the product of the matrices of  $A$  and  $B$ .
- (c) Find the matrices of  $D^k$  for  $k = 3, 4, \dots$ . Do you get the zero operator  $\mathbf{O}$  for some value of  $k$ ? An operator  $A$  for which it happens that  $A^k = \mathbf{O}$  for some  $k$  is called a *nilpotent* operator.

**Problem 5.** Let  $V_n(a, b; w(x))$  stand for the linear space of polynomials of degree no greater than  $n$  endowed with the inner product

$$\langle P, Q \rangle = \int_a^b P(x) Q(x) w(x) dx .$$

We want to construct polynomials  $D_0, D_1, \dots, D_n$  satisfying the following conditions:

- (i) the polynomial  $D_k$  is of degree  $k$ ;
- (ii) then coefficient of  $x^k$  in  $D_k$  is equal to 1 (such polynomials are called *monic*);
- (iii) the polynomials  $D_0, D_1, D_2, \dots, D_n$  form an orthogonal basis in the space of polynomials  $V_n(0, \infty; w(x) = e^{-x})$ .

In the solution of this problem the following identity will be handy:

$$\int_0^\infty x^k e^{-x} dx = k!$$

(where, by definition,  $0! = 1$ ).

- (a) Clearly,  $D_0(x) = 1$  for each  $x \in [0, \infty)$ . Find the only monic polynomial  $D_1$  of degree 1 that is orthogonal to  $D_0$ .
- (b) Find the only monic quadratic polynomial  $D_2$  that is orthogonal to both  $D_0$  and  $D_1$ .
- (c) Show that the polynomial  $P(x) = x^2 + 3$  can be represented as a linear combination of the polynomials  $D_0, D_1$  and  $D_2$  as follows:  $P = D_2 + 4D_1 + 5D_0$ .
- (d) Show by direct integration that  $\langle D_0, D_0 \rangle = 1$ ,  $\langle D_1, D_1 \rangle = 1$ ,  $\langle D_2, D_2 \rangle = 4$ .
- (e) Find the orthogonal projection,  $\text{proj}_{D_0+2D_1} Q$ , of the polynomial  $Q(x) = x^2 + 3$  onto the “straight line”

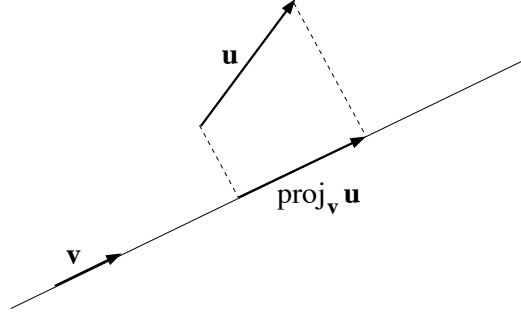
$$\ell := \{t(D_0 + 2D_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space  $V_2(0, \infty; e^{-x})$ . If you have solved part (c), then finding this orthogonal projection should be easy.

*Hint:* If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in the inner product linear space  $V$ , then the orthogonal projection of the vector  $\mathbf{u}$  onto the straight line in the direction of  $\mathbf{v}$  is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

– see the picture below.



- (f) Finally, let  $\tilde{D}_k := \mu_k D_k$ , where  $\mu_k > 0$  is a constant (depending on  $k$ ) such that the norm,

$$\|\tilde{D}_k\| := \sqrt{\langle \tilde{D}_k, \tilde{D}_k \rangle},$$

of the polynomial  $\tilde{D}_k$  is 1. Find the explicit expressions for  $\tilde{D}_0(x)$ ,  $\tilde{D}_1(x)$ , and  $\tilde{D}_2(x)$ .

**Problem 6.** As we mentioned in class, one can define different norms in the same linear space. In this problem you will study different norms in  $\mathbb{R}^2$ . Let  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \mathbf{i} + u_2 \mathbf{j} \in \mathbb{R}^2$ .

- (a) Define the norm  $\|\mathbf{u}\|_2$  by

$$\|\mathbf{u}\|_2 := \sqrt{u_1^2 + u_2^2}.$$

Draw the unit disk in  $\mathbb{R}^2$  in this norm, i.e., the set of vectors  $\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_2 \leq 1\}$ .

- (b) Define the norm  $\|\mathbf{u}\|_1$  by

$$\|\mathbf{u}\|_1 := |u_1| + |u_2|.$$

Draw the unit disk in  $\mathbb{R}^2$  in this norm, i.e., the set of vectors  $\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_1 \leq 1\}$ .

- (c) Define the norm  $\|\mathbf{u}\|_\infty$  by

$$\|\mathbf{u}\|_\infty := \max\{|u_1|, |u_2|\}.$$

Draw the unit disk in  $\mathbb{R}^2$  in this norm, i.e., the set of vectors  $\{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\|_\infty \leq 1\}$ .

- (d) Two norms  $\|\cdot\|$  and  $\|\cdot\|'$  on the same linear space are said to be *equivalent* if there exist positive constants  $C_1$  and  $C_2$  such that  $C_1 \|\mathbf{u}\| \leq \|\mathbf{u}\|' \leq C_2 \|\mathbf{u}\|$  for any vector  $\mathbf{u} \in V$ . In class we proved that  $\|\mathbf{u}\|_\infty \leq \|\mathbf{u}\|_2 \leq \sqrt{2} \|\mathbf{u}\|_\infty$ , i.e., that the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are equivalent.

Show that the norms  $\|\mathbf{u}\|_1$  and  $\|\mathbf{u}\|_\infty$  are equivalent (you have to find the corresponding constants  $C_1$  and  $C_2$ ).

- (e) Use the fact (proved in class) that the norms  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  are equivalent and the fact (proved in part (d)) that the norms  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are equivalent to prove that the norms  $\|\mathbf{u}\|_1$  and  $\|\mathbf{u}\|_2$  are equivalent.