MATH 5453 Homework 6a <u>Not</u> Due Tue, Oct 14

Problem 12 from Section 2.2 of the book.

Additional problem 1. Consider the measure space $(\mathbb{R}, \mathcal{L}, m)$ (where m is the Lebesgue measure). Let $f : [0,\infty) \to [0,\infty)$ be the function $f(x) = \sqrt{x}$. For $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots, 2^{2n} - 1$, define the sets

$$E_{nk} := f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) , \qquad F_n := f^{-1}\left((2^n, \infty]\right) ,$$

as in the proof of Theorem 2.10(a) from the book. Define the sequence $\{\phi_n\}_{n=1}^{\infty}$ of simple functions

$$\phi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_{nk}} + 2^n \chi_{F_n} .$$

According to Theorem 2.10(a), this sequence satisfies $0 \le \phi_1 \le \phi_2 \le \cdots \le f$, converges to f pointwise, and converges to f uniformly on any set on which f is bounded.

(a) Compute $\int_0^1 \phi_n(x)\, \mathrm{d}m(x) := \int_{[0,1]} \phi_n(x)\, \mathrm{d}m(x) := \int \phi_n(x) \chi_{[0,1]}(x)\, \mathrm{d}m(x)$.

Hint: You may use that

$$\sum_{j=0}^{m} j = \frac{1}{2} m(m+1) , \qquad \sum_{j=0}^{m} j^2 = \frac{1}{6} m(m+1)(2m+1) .$$

(b) Using the definition of the integral of non-negative functions

$$\int f \, \mathrm{d}\mu := \sup\left\{ \int \phi \, \mathrm{d}\mu \ : \ 0 \le \phi \le f \ , \ \phi \text{ simple} \right\} ,$$

to find $\int_0^1 f(x)\, \mathrm{d}m(x)$. Explain which theoretical result allows you to find this easily.

Additional problem 2. Repeat all parts of Additional problem 1, but instead of using the Lebesgue measure m, use the Lebesgue-Stieltjes measure μ_G corresponding to the (non-decreasing right-continuous) function $G : [0,\infty) \to \mathbb{R}$ given by $G(x) = \sqrt{x}$.

Additional problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, a be an arbitrary real number, and let the measure μ_F corresponds to the (non-decreasing right-continuous) function $F = \chi_{[a,\infty)} : \mathbb{R} \to \mathbb{R}$. Directly from the definitions, compute $\int f \, \mathrm{d}\mu_F$.

Additional problem 4. Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space, f be an arbitrary integrable function, and $t > 0$ be an arbitrary positive number. Prove *Markov's inequality*,

$$\mu\left(\{x \in X : |f(x)| \geq t\}\right) \leq \frac{1}{t} \int |f| \, \mathrm{d}\mu \ .$$

Hint: Think about the indicator function of the set $\{x \in X : |f(x)| \geq t\}$. What can you say about the values of f on this set and on its complement?

Additional problem 5. Let E and F be subsets of a set X. Prove the following identities about indicator functions:

$$\chi_{E^c} = 1 - \chi_E \ , \qquad \chi_{E \cap F} = \chi_E \cdot \chi_F \ ,$$

and find expressions for $\chi_{E \cup F}$ and $\chi_{E \triangle F}$ in terms of χ_E and χ_F.