

MATH 5453 Homework 6a Not Due Tue, Oct 14

Problem 12 from Section 2.2 of the book.

Additional problem 1. Consider the measure space $(\mathbb{R}, \mathcal{L}, m)$ (where m is the Lebesgue measure). Let $f : [0, \infty) \rightarrow [0, \infty)$ be the function $f(x) = \sqrt{x}$. For $n = 0, 1, 2, \dots$ and $k = 0, 1, 2, \dots, 2^{2^n} - 1$, define the sets

$$E_{nk} := f^{-1} \left(\left(\frac{k}{2^n}, \frac{k+1}{2^n} \right] \right) , \quad F_n := f^{-1}((2^n, \infty]) ,$$

as in the proof of Theorem 2.10(a) from the book. Define the sequence $\{\phi_n\}_{n=1}^\infty$ of simple functions

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} \frac{k}{2^n} \chi_{E_{nk}} + 2^n \chi_{F_n} .$$

According to Theorem 2.10(a), this sequence satisfies $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f$, converges to f pointwise, and converges to f uniformly on any set on which f is bounded.

(a) Compute $\int_0^1 \phi_n(x) dm(x) := \int_{[0,1]} \phi_n(x) dm(x) := \int \phi_n(x) \chi_{[0,1]}(x) dm(x)$.

Hint: You may use that

$$\sum_{j=0}^m j = \frac{1}{2}m(m+1) , \quad \sum_{j=0}^m j^2 = \frac{1}{6}m(m+1)(2m+1) .$$

(b) Using the definition of the integral of non-negative functions

$$\int f d\mu := \sup \left\{ \int \phi d\mu : 0 \leq \phi \leq f , \phi \text{ simple} \right\} ,$$

to find $\int_0^1 f(x) dm(x)$. Explain which theoretical result allows you to find this easily.

Additional problem 2. Repeat all parts of Additional problem 1, but instead of using the Lebesgue measure m , use the Lebesgue-Stieltjes measure μ_G corresponding to the (non-decreasing right-continuous) function $G : [0, \infty) \rightarrow \mathbb{R}$ given by $G(x) = \sqrt{x}$.

Additional problem 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, a be an arbitrary real number, and let the measure μ_F corresponds to the (non-decreasing right-continuous) function $F = \chi_{[a, \infty)} : \mathbb{R} \rightarrow \mathbb{R}$. Directly from the definitions, compute $\int f d\mu_F$.

Additional problem 4. Let (X, \mathcal{M}, μ) be an arbitrary measure space, f be an arbitrary integrable function, and $t > 0$ be an arbitrary positive number. Prove *Markov's inequality*,

$$\mu(\{x \in X : |f(x)| \geq t\}) \leq \frac{1}{t} \int |f| d\mu .$$

Hint: Think about the indicator function of the set $\{x \in X : |f(x)| \geq t\}$. What can you say about the values of f on this set and on its complement?

Additional problem 5. Let E and F be subsets of a set X . Prove the following identities about indicator functions:

$$\chi_{E^c} = 1 - \chi_E , \quad \chi_{E \cap F} = \chi_E \cdot \chi_F ,$$

and find expressions for $\chi_{E \cup F}$ and $\chi_{E \Delta F}$ in terms of χ_E and χ_F .