MATH 5763 **Homework 6** **Due Tue, 3/5/2019**

Problem 1. Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate λ. You start observing the Poisson process equipped with a counter (which counts the events that have occurred) and a stopwatch that will give you the times T_n of occurrence of the events of the process. You set the counter to 0, and at time 0 start the stopwatch. However, right after you started the observation, you had to leave the room. When you came back, you looked at the stopwatch which showed time t, and at the event counter which showed 1. Therefore, you know that in the time interval $(0, t]$ exactly one event has occurred, i.e., $N_t = 1$. You don't know T_1, so you can think of it as a random variable; clearly, it should be a continuous random variable whose distribution is nonzero only on the interval $(0, t]$. Find the distribution of T_1 given that you know that $N_t = 1$.

Hint: Consider the conditional probability $\mathbb{P}(T_1 \leq s|N_t = 1)$ for $0 < s \leq t$ and use some of the basic properties of the Poisson process. You may use the fact that N_t is a Poisson random variable with parameter λt. Please specify what properties of the Process you are using at each step. Given the properties of the Poisson process, the answer to this problem should be intuitively obvious (but you have to obtain it mathematically). Your solution should be a couple of lines.

Problem 2. Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate λ. Let $0 < s < t$. If you know that exactly n events occurred by time t (i.e., that $N_t = n$), prove that the random variable N_s (the number of events that have occurred by time s) has binomial distribution with parameters n and $\frac{s}{t}$. A fancy way of writing this is $N_s \sim \mathrm{Bin}(N_t, \frac{s}{t})$. You may use any fact about the Poisson process that we have mentioned, but please write specifically what you are using.

Problem 3. Let $N = \{N_t : t \geq 0\}$ be a (time-homogeneous) Poisson process with rate λ. Define the *flip-flop process* $X = \{X_t : t \geq 0\}$ with state space $S = \{0, 1\}$ by

$$X_t = \frac{1}{2} + (-1)^{N_t}\left[X_0 - \frac{1}{2}\right] ,$$

where X_0 is a random variable with values in S; assume that X_0 is independent of the Poisson process N. In other words, the flip-flop process switches between the states 0 and 1 at each event of N; this is a particular case of the flip-flop process studied in class, because the probability of jumping from 0 to 1 is the same as the probability of jumping from 1 to 0. Since N is a Markov process, X is also a Markov process. Let $\mathbf{P}_t = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix}$ be the stochastic semigroup of the process X, and $\mathbf{G}$ be the generator of $\mathbf{P}_t$.

(a) Find the short-time transition probabilities $p_{ij}(h) = \mathbb{P}(X_{t+h} = j|X_t = i)$ and show that the generator of the stochastic process X is $\mathbf{G} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$.

(b) To find the time evolution of the chain X (i.e., to find the stochastic semigroup $\mathbf{P}_t = e^{t\mathbf{G}}$), one needs to find $\mathbf{G}^n$ for $n \in \mathbb{N}$ (of course, $\mathbf{G}^0 = \mathbf{I}$, the identity matrix).

One way to compute $\mathbf{G}^n$ is to first try to diagonalize it by a similarity transformation, $\widetilde{\mathbf{G}} = \mathbf{M}^{-1}\mathbf{G}\mathbf{M}$. Then compute the nth power of the diagonal matrix $\widetilde{\mathbf{G}}$ (which is very easy), and finally use that $\widetilde{\mathbf{G}}^n = \mathbf{M}^{-1}\mathbf{G}^n\mathbf{M}$, so that $\mathbf{G}^n = \mathbf{M}\widetilde{\mathbf{G}}^n\mathbf{M}^{-1}$. In fact, one can directly compute the diagonal matrix $\mathrm{e}^{t\widetilde{\mathbf{G}}}$, and then to use that

$$\mathbf{P}_t = \mathrm{e}^{t\mathbf{G}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathbf{G}^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\mathbf{M}\widetilde{\mathbf{G}}^n\mathbf{M}^{-1} = \mathbf{M}\sum_{n=0}^{\infty} \frac{t^n}{n!}\,\widetilde{\mathbf{G}}^n\mathbf{M}^{-1} = \mathbf{M}\,\mathrm{e}^{t\widetilde{\mathbf{G}}}\mathbf{M}^{-1}\ .$$

You may use that in this problem one can take $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{M}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

At the end, you should obtain that $\mathbf{P}_t = \begin{pmatrix} \frac{1}{2}(1+\mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1-\mathrm{e}^{-2\lambda t}) \\ \frac{1}{2}(1-\mathrm{e}^{-2\lambda t}) & \frac{1}{2}(1+\mathrm{e}^{-2\lambda t}) \end{pmatrix}$, but I would like to see the details of your computations.

(c) Now you will find $\mathbf{P}_t$ directly, without using the generator $\mathbf{G}$. (Of course, you have to pretend that you don't know the answer.) One can do this using several methods.

The standard method is to solve, say, the Kolmogorov backward equations, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}_t = \mathbf{G}\mathbf{P}_t$, with appropriate initial conditions (the Kolmogorov forward equations can also be used). You do not need to do it here because we did this in class (see also the example on pages 129–131 of Lefevbre's book).

A trickier method for computing $\mathbf{P}_t$ (which works in this particular problem) is the following. Note that $p_{01}(t) = \mathbb{P}(X_{s+t} = 1 \,|\, X_s = 0)$ is equal to the probability that there were an odd number of events of the Poisson process N of intensity λ in the interval $(s, s+t]$. Using the explicit expression for the probability of exactly k events of a Poisson process to occur in a time interval of length t, compute $p_{01}(t)$.

From $p_{01}(t)$, one can easily find $p_{00}(t)$ (how?), and the values of $p_{10}(t)$, and $p_{11}(t)$ can be obtained simply by relabeling, but you do *not* need to do this here.

Hint: Note that $\displaystyle\sum_{j\ \text{odd}} \frac{\alpha^j}{j!} = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!} = \frac{1}{2}\left(\mathrm{e}^{\alpha} - \mathrm{e}^{-\alpha}\right)$.

(d) Find the stationary distribution $\boldsymbol{\pi}$ by using the generator $\mathbf{G}$.

(e) Now assume that initially the chain X is in state 0 (i.e., that $X_0 = 0$). Determine the probability distribution $\mathbf{p}(t) = (p_0(t)\ \ p_1(t))$ (where $p_j(t) = \mathbb{P}(X_t = j)$) of the chain X at time t by using your results above. As t goes to infinity, does $\mathbf{p}(t)$ tend to the stationary distribution $\boldsymbol{\pi}$?

(f) Define the generating functions $G_i(\xi, t) := \displaystyle\sum_{j=0}^{1} p_{ij}(t)\,\xi^j$, and show that G_i satisfies the first-order partial differential equation $\frac{\partial G_i}{\partial t} + 2\lambda(\xi - 1)\frac{\partial G_i}{\partial \xi} = \lambda(\xi - 1)$. Since the proofs for G_0 and G_1 are essentially the same, give a proof only for G_0. You have to do this by using the Kolmogorov forward equations, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}_t = \mathbf{P}_t\mathbf{G}$, i.e.,

$$\begin{pmatrix} p_{00}'(t) & p_{01}'(t) \\ p_{10}'(t) & p_{11}'(t) \end{pmatrix} = \begin{pmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{pmatrix} \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}\ .$$

(g) What are the initial conditions that G_0 and G_1 must satisfy? Why? (Recall that we assumed that initially the chain is in state 0.)

One can show that the solution of the PDE above and the corresponding initial conditions is

$$G_0(\xi,t) = \frac{1}{2}\left[1+\xi+(1-\xi)\mathrm{e}^{-2\lambda t}\right] , \qquad G_1(\xi,t) = \frac{1}{2}\left[1+\xi-(1-\xi)\mathrm{e}^{-2\lambda t}\right] .$$

(h) From the very definition of $G_i(\xi,t)$, one can show that

$$\mathbb{E}[X_t|X_0=i] = \frac{\partial G_i}{\partial \xi}(1,t)$$

and

$$\mathrm{Var}[X_t|X_0=i] = \mathbb{E}[X_t^2|X_0=i] - \mathbb{E}[X_t|X_0=i]^2 = \frac{\partial^2 G_i}{\partial^2 \xi}(1,t) + \frac{\partial G_i}{\partial \xi}(1,t) - \left[\frac{\partial G_i}{\partial \xi}(1,t)\right]^2 .$$

This is completely analogous to what you did in Problem 3 of Homework 5, so you do *not* need to do this here.

Use the concrete expressions for the generating functions $G_i(\xi,t)$ of the flip-flop problem (written in part (g)) in order to find the conditional expectation $\mathbb{E}[X_t \,|\, X_0 = 0]$ and the conditional variance $\mathrm{Var}[X_t \,|\, X_0 = 0]$, and sketch $\mathbb{E}[X_t \,|\, X_0 = 0]$ and $\mathrm{Var}[X_t \,|\, X_0 = 0]$ as functions of t.

(i) Without looking at your results obtained in part (h), guess the values of $\mathbb{E}[X_t \,|\, X_0 = 0]$ and $\mathrm{Var}[X_t \,|\, X_0 = 0]$ in the limiting cases $t \to 0^+$ and $t \to \infty$ (the case $t \to \infty$ will require some very simple calculations using the result of part (d)). Give some simple reasoning to justify your guesses.

Check that the expressions for $\mathbb{E}[X_t \,|\, X_0 = 0]$ and $\mathrm{Var}[X_t \,|\, X_0 = 0]$ obtained in part (h) matches your intuitive expectation as $t \to 0^+$ and $t \to \infty$.

Problem 4. A simple symmetric random walk starts at position 1 and returns to position 1 at time $2n$ (i.e., $X_0 = 1$, $X_{2n} = 1$). If we use the pictorial representation of a random walk in the "time-position" plane (in which the random walk is represented as as a connected graph like in Food for Thought Problem 2 of Homework 5), then we can say that the random walk starts at the point $(0,1)$ and goes to $(2n,1)$ after $2n$ steps.

(a) Using the reflection principle from Food for Thought Problem 2 of Homework 5, show that there are $\dfrac{(2n)!}{n!(n+1)!}$ different paths between the points $(0,1)$ and $(2n,1)$ that do not ever visit the origin (by "origin" we mean "position 0").

(b) What is the probability that the walk starting at $(0,1)$ ends at $(2n,1)$ after $2n$ steps without ever visiting the origin, assuming that the random walk is symmetric?

(c) Show that the probability that the first visit to the origin occurs at time $2n+1$ is

$$p_n = \frac{1}{2^{2n+1}} \frac{(2n)!}{n!(n+1)!} .$$