# MATH 5763 Homework 6 Due Thu, Feb 28

**Problem 1.** Let $X$ be a geometric random variable with parameter $p \in (0, 1)$, i.e., with p.m.f.

$$p_X(k) = P(X = k) = (1 - p)^{k-1} p , \qquad k \in \mathbb{N} = \{1, 2, 3, \dots\} .$$

(a) From the p.m.f. of $X$, find the probability that $X > n$ for $n \in \mathbb{N}$.

(b) Using your result from (a), show that the geometric random variable is *memoryless*, i.e., that it satisfies the property $P(X > n + m | X > n) = P(X > m)$ for any $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

(c) Recall the meaning of the geometric random variables, and explain the reason for calling the property you proved in part (b) "memorylessness" (i.e., lack of memory).

(d) Can you explain the the reason for the memorylessness of geometric random variables intuitively, without any math?

**Problem 2.** Let the transition matrix of a Markov chain with two states be $\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$, where $a$ and $b$ are numbers between 0 and 1.

(a) Show that $\mathbf{P}^n = \dfrac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \dfrac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}$.

*Hint:* One way to do this is the following. Let $\alpha_1$ and $\alpha_2$ be the roots of the quadratic equation $\det(\mathbf{P} - \alpha\mathbf{I}) = 0$ (where $\mathbf{I}$ is the unit $2 \times 2$ matrix). In this problem you should obtain $\alpha_1 = 1$, $\alpha_2 = 1 - a - b$. If $\alpha_1 \neq \alpha_2$ (as in this problem), then $\mathbf{P}^n = \mathbf{C}_1\, \alpha_1^n + \mathbf{C}_2\, \alpha_2^n$, where $\mathbf{C}_1$ and $\mathbf{C}_2$ are $2 \times 2$ matrices whose entries can be found from

$$\mathbf{P}^0 = \mathbf{C}_1 + \mathbf{C}_2 = \mathbf{I} , \qquad \mathbf{P}^1 = \mathbf{C}_1\, \alpha_1 + \mathbf{C}_2\, \alpha_2 = \mathbf{P} .$$

(b) Find $\lim\limits_{n\to\infty} \mathbf{P}^n$.

(c) Use your result from part (b) to find the stationary distribution $\boldsymbol{\pi}$ of this Markov chain.

**Problem 3.** Let $N = \{N_t : t \geq 0\}$ be a Poisson process of intensity $\lambda$. We showed in class that the number of events occurring in the interval $(0, t]$ is

$$P(N_t = j) = \frac{(\lambda t)^j}{j!}\, \mathrm{e}^{-\lambda t} , \qquad j \in \mathbb{Z}_+ = \{0, 1, 2, 3, \dots\} .$$

(a) What type of random variable is $N_t$? What are $E[N_t]$ and $\operatorname{Var} N_t$?

*Hint:* Use the property of this type of random variables "for free" (i.e., you don't need to do any computations).

(b) If $s < t$ and $i < j$, compute the probability of the event $\{N_t = j\}$ given that the event $\{N_s = i\}$ has occurred.

*Hint:* In other words, you have to compute the conditional probability $P(N_t = j|N_s = i)$. This can be done without *any* calculation if you just think about the meaning of $N_t$! But I do want to see your reasoning!

(c) Again, let $s < t$ and $i < j$. Show that the probability of the event $\{N_t = j\} \cap \{N_s = i\}$ is equal to

$$P(N_t = j, N_s = i) = \frac{\lambda^j}{i!(j-i)!} (t-s)^{j-i} s^i \mathrm{e}^{-\lambda t} .$$

*Hint:* Note that $P(N_t = j, N_s = i) = P(N_t = j|N_s = i)\, P(N_s = i)$, and use your result from part (b).

**Problem 4.** A *death process* is a random process that describes the number of people in a society where the only reason for changing the number of people is dying (nobody is born, there is no immigration, etc.). We say that the random process $X = \{X_t : t \geq 0\}$ is a death process with parameter $\mu$ if each person dies independently of every other person, and the probability that each person dies in one unit of time is $\mu$ (we assume that the units of time we use are much shorter than the average lifetime of the people). Clearly, the probability of a death of a person in one unit of time in a population of $i$ people is $i\mu$ (again, we assume that the unit of time is "short").

Here is the precise mathematical definition of a death process with parameter $\mu$:

- the state space of the process is $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$;
- the process is non-increasing, i.e., if $s < t$, then $X_s \geq X_t$;
- if $h$ is a very small positive number, then, for $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$P(X_{t+h} = j|X_t = i) = \begin{cases} i\mu h + o(h) & \text{if } j = i-1 , \\ 1 - i\mu h + o(h) & \text{if } j = i , \\ o(h) & \text{if } j > i \text{ or } j \leq i-2 , \end{cases}$$

  and

$$P(X_{t+h} = 0|X_t = i) = \begin{cases} \mu h + o(h) & \text{if } i = 1 , \\ 1 & \text{if } i = 0 , \\ o(h) & \text{if } i > 1 . \end{cases}$$

  Here $o(h)$ is a function satisfying $\lim\limits_{h\to 0} \frac{o(h)}{h} = 0$;
- for $s < t$, the difference $X_t - X_s$ (equal to the number of deaths in the interval $(s,t]$) does not depend on what has happened in the time interval $(0, s]$.

In this problem you will analyze some aspects of this process.

(a) Let $p_i(t) = P(X_t = i)$. Condition on $X_t$ to derive the equations

$$\begin{aligned} p_0(t+h) &= \mu h p_1(t) + p_0(t) + o(h) \ , \\ p_j(t+h) &= (j+1)\mu h p_{j+1}(t) + (1 - j\mu h) p_j(t) + o(h) \ , \qquad j \in \mathbb{N} \ . \end{aligned}$$

(b) Subtract $p_j(t)$ from the $j$th equation from (a), divide through by $h$, and take the limit $h \to 0$, to obtain the system

$$\begin{aligned} p_0'(t) &= \mu p_1(t) \ , \\ p_j'(t) &= (j+1)\mu p_{j+1}(t) - j\mu p_j(t) \ , \qquad j \in \mathbb{N} \ . \end{aligned}$$

Let the initial condition be $X_0 = I$, where $I$ is some positive integer.

(c) Define the *generating function*

$$\Delta(\xi, t) := \sum_{j=0}^{\infty} p_j(t)\, \xi^j$$

and show that

$$\frac{\partial \Delta}{\partial \xi} = \sum_{j=0}^{\infty} j\, p_j(t)\, \xi^{j-1} = \sum_{j=1}^{\infty} j\, p_j(t)\, \xi^{j-1} \ , \qquad \frac{\partial \Delta}{\partial t} = \sum_{j=0}^{\infty} p_j'(t)\, \xi^j \ .$$

(d) Use the differential equations from part (b) to show that $\Delta(\xi, t)$ of the death process satisfies the partial differential equation

$$\frac{\partial \Delta}{\partial t} = \mu(1 - \xi) \frac{\partial \Delta}{\partial \xi} \ ,$$

and the initial condition $\Delta(\xi, 0) = \xi^I$ (where $I = X_0$ is the initial population).

*Hint:* Multiply the differential equation for $p_j'(t)$ by $\xi^j$ and add all the equations.

(e) How can the probabilities $p_j(t) = P(X_t = j)$ be expressed in terms of $\xi$-derivatives of $\Delta(\xi, t)$ evaluated at $\xi = 0$? Use this to find the explicit expressions for $P(X_t = 0)$ and $P(X_t = 1)$, using that the solution of the initial-value problem for the generating function posed in part (d) is

$$\Delta(\xi, t) = \left[1 + (\xi - 1)\, \mathrm{e}^{-\mu t}\right]^I$$

(there is no need to derive this expression).

(f) Show that

$$\frac{\partial \Delta}{\partial \xi}(1, t) = E[X_t] \ ,$$

and use this fact to find $E[X_t]$ for the death process.

(g) Using the same idea as in part (f), express the variance of $X_t$ in terms of derivatives of its generating function evaluated at $\xi = 1$. Use the explicit expression for $\Delta(\xi, t)$ given in (e) to find $\operatorname{Var} X_t$ for the death process.

(h) Radioactive decay is an example of a death process, if we think of a nucleus of the radioactive isotope as “alive” before it decays, and “dead” after that. The “half-life”, $T_{1/2}$, of a radioactive isotope is defined as the time after which only half of the initial number of nuclei of this isotope are “alive”. How is $T_{1/2}$ related to $\mu$?