MATH 5483 Homework 6 Due Thu, Feb 28

Problem 1. Let $\mathcal{L} := (C^0([-1,1]), \| \ \|)$ be the normed space of continuous functions on $[-1,1]$ with norm defined by

$$\|f\| := \int_{-1}^{1} |f(x)| \, \mathrm{d}x \ .$$

Consider the sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ defined by

$$f_n(x) := \begin{cases} 0 & \text{if } x \in [-1,0] \ , \\ nx & \text{if } x \in [0, \frac{1}{n}] \ , \\ 1 & \text{if } x \in [\frac{1}{n}, 1] \ . \end{cases}$$

(a) Sketch the graph of f_n.

(b) For a pair of natural numbers m and n, find $\|f_m - f_n\|$.

(c) Show that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

(d) Which function would be a candidate for a limit of $\{f_n\}_{n\in\mathbb{N}}$ as $n \to \infty$? Does this candidate for a limit belong to the space $\mathcal{L}$?

(e) Is $\mathcal{L}$ a complete normed space?

Problem 2. Let $\{g_n\}_{n\in\mathbb{Z}}$ be any system of L^2 functions. Show that span $\{g_n\}_{n\in\mathbb{Z}}$ is a linear space (i.e., that it is closed under the formation of finite linear combination).

Problem 3. Let g be an L^2 function on $\mathbb{R}$.

(a) Show that

$$\sum_{n\in\mathbb{N}} |\widehat{g}(\gamma + n)|^2 = \sum_{n\in\mathbb{N}} \langle g, T_n g\rangle \, \mathrm{e}^{-2\pi i n\gamma} \ .$$

(b) Conclude that if g has compact support, then $\sum_{n\in\mathbb{N}} |\widehat{g}(\gamma + n)|^2$ is a trigonometric polynomial.

Problem 4. Consider the collection of functions (orthonormal or not) $\{T_n g\}_{n\in\mathbb{Z}}$ of translates of the function $g \in L^2(\mathbb{R})$. Show that $\overline{\text{span}}\,\{T_n g\}_{n\in\mathbb{Z}}$ is invariant under integer translation. In other words, show that if $f \in \overline{\text{span}}\,\{T_n g\}_{n\in\mathbb{Z}}$, then $T^k f \in \overline{\text{span}}\,\{T_n g\}_{n\in\mathbb{Z}}$ for any $k \in \mathbb{Z}$.