MATH 5463 Homework 6 Due Thu, Apr 2

Problem 1. Let K be the operator taking each $f \in L^2([0,1])$ to the function $Kf : [0,1] \to \mathbb{C}$ that is defined by

$$(Kf)(x) = \frac{1}{x^{4/3}} \int_0^x f(t)\,\mathrm{d}t\ , \quad x \in [0,1]\ .$$

(a) Show that $\|Kf\|_1 \le C\|f\|_2$, where C is a constant independent of f.

Hint: Write $\|Kf\|_1 = \int_0^1 \left| \frac{1}{x^{4/3}} \int_0^x f(t)\,\mathrm{d}t \right| \mathrm{d}x \le \int_0^1 \int_0^x \frac{1}{x^{4/3}} |f(t)|\,\mathrm{d}t\,\mathrm{d}x$, then change the order of integration (checking that you are allowed to do that), perform one of the integrations explicitly, and in the remaining integral apply Hölder's inequality with $p = q = \frac{1}{2}$. You may need to use that $\int_0^1 (t^{-1/3} - 1)^2\,\mathrm{d}t = 1$.

(b) Interpret the result of (a) as a statement about the domain and the range of the operator K.

Problem 2. Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing right-continuous function such that $F(\infty) - F(-\infty) = 1$, which implies that the corresponding (positive) Borel measure μ_F is a probability measure on $\mathbb{R}$ (i.e., that $\mu_F(\mathbb{R}) = 1$). Let m be the Lebesgue measure on $\mathbb{R}$, and $c > 0$ be an arbitrary constant. Prove that

$$\int_{\mathbb{R}} [F(x+c) - F(x)]\ \mathrm{d}m(x) = c\ .$$

Hint: Write down $F(x+c) - F(x)$ as the measure μ_F of a some subset of $\mathbb{R}$, then represent the resulting integral as a double integral, and use Tonelli-Fubini theorem to change the order of integration. It will be useful to note that

$$\chi_{(x,x+c]}(y) = \chi_{\{x<y\le x+c\}}(x,y) = \chi_{[y-c,y)}(x)\ .$$

Problem 3. Let $X = \mathbb{R}\times\mathbb{R}_{\mathrm{d}}$, where $\mathbb{R}_{\mathrm{d}}$ stands for the set of all real numbers with the discrete topology (i.e., each subset of $\mathbb{R}_{\mathrm{d}}$ is open). For $f : X \to \mathbb{C}$ and $E \subset X$, let $f^y(x) := f(x,y)$ and $E^y := \{x \in \mathbb{R} : (x,y) \in E\}$ (as in Section 2.5 on product measures).

(a) What are the compact sets in $\mathbb{R}_{\mathrm{d}}$?

(b) Prove that $f \in C_c(X)$ if and only if $f^y \in C_c(\mathbb{R})$ for all $y \in \mathbb{R}_{\mathrm{d}}$ and $f^y = 0$ for all but finitely many $y \in \mathbb{R}_{\mathrm{d}}$.

(c) Define a positive linear functional on $C_c(X)$ by

$$I(f) = \sum_{y \in \mathbb{R}_\mathrm{d}} \int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \ ,$$

and let μ be the associated Radon measure on X. Then $\mu(E) = \infty$ for any $E \subset X$ with $E^y \neq \emptyset$ for uncountably many $y \in \mathbb{R}_\mathrm{d}$.

(d) Let $E = \{0\} \times \mathbb{R}_\mathrm{d}$. Then $\mu(E) = \infty$, but $\mu_K = 0$ for all compact $K \subset E$.

(e) What does the fact proved in (d) say about whether the Radon measure μ is regular?

(e) How does your observation in (e) go together with Proposition 7.5 and Corollary 7.6?