MATH 5453 Homework 6 Due Thu, Oct 9

Problem 16 from Section 2.2 of the book.

Additional problem 1.

(a) Construct a sequence of functions in L^+ such that $f_n \geq f_{n+1}$ a.e. for all n such that $\lim_{n\to\infty} f_n = 0$ a.e., but $\lim_{n\to\infty} \int f_n \neq 0$.

Remark: This shows that there is no direct analog of the Monotone Convergence Theorem for decreasing sequences of functions in L^+, unless some additional condition is imposed.

(b) Use the Monotone Convergence Theorem (and Corollary 2.17) to show that, if $\{f_n\}$ is a sequence in L^+ such that $f_n \geq f_{n+1}$ a.e. for all n, $\int f_1 < \infty$, and $f := \lim_{n\to\infty} f_n (= \inf_n f_n)$, then $\int f = \lim_{n\to\infty} \int f_n$.

Additional problem 2. In all parts of this problem, m stands for the Lebesgue measure.

(a) Let $f_n := \chi_{(n,n+1)} : \mathbb{R} \to \mathbb{R}$. Show that $f_n \to 0$ pointwise and find $\int f_n \, \mathrm{d}m$. Does your result contradict the Monotone Convergence Theorem?

(b) Let $f_n := n\chi_{(0,1/n)} : [0,1] \to \mathbb{R}$. Find the pointwise limit of this function sequence and compute $\int f_n \, \mathrm{d}m$. Explain why this example does not contradict the Monotone Convergence Theorem.

(c) Construct an example of a sequence of measurable functions $\{f_n\}$ defined on $[0,1]$ and taking values in $[0,\infty]$, such that

$$\int \liminf f_n \, \mathrm{d}m < \liminf \int f_n \, \mathrm{d}m \ .$$

Additional problem 3.

(a) Let ϵ be an arbitrary number in $(0,1)$. Carry out the construction of the generalized Cantor set (see page 30 of Folland's book) to construct such a set with Lebesgue measure at least $1-\epsilon$.

(b) Use your result from part (a) to prove that for every $\epsilon \in (0,1)$ there exists an open dense subset $V \subset \mathbb{R}$ whose Lebesgue measure does not exceed ϵ.

Additional problem 4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and

$$E := \left\{ x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \ \text{ exists in } \mathbb{R} \right\} .$$

(note that we might have $E = \emptyset$ because we do not make any specific assumption about the pointwise convergence of $\{f_n\}_{n=1}^{\infty}$). Prove that E is a countable intersection of countable unions of closed sets.

Hint: Recall that, by definition, a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is Cauchy if for each ϵ there exists a number N such that $m, n > N$ implies that $|a_m - a_n| < \epsilon$ (or, equivalently, if $m, n > N$ implies $|a_m - a_n| \le \epsilon$). Use that for $x \in \mathbb{R}$, the sequence of real numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges if and only if it is Cauchy. Try to write the statement "$\{f_n(x)\}_{n=1}^{\infty}$ is Cauchy" as an equivalent statement involving the membership of x in some set-theoretic combination of the sets

$$E_{ijk} = \left\{ y \in \mathbb{R} : |f_i(y) - f_j(y)| \le \frac{1}{k} \right\} ,$$

where i, j and k are natural numbers. Define the family of functions $\{g_{ij}\}_{i,j=1}^{\infty}$ by $g_{ij}(x) := |f_i(x) - f_j(x)|$. Are these functions continuous? How can you write the sets E_{ijk} in terms of the functions g_{ij}? What do you know about the pre-images of closed sets under a continuous function? Finally, what do you know about intersection of closed sets?