

**Problem 16** from Section 2.2 of the book.

**Additional problem 1.**

- (a) Construct a sequence of functions in  $L^+$  such that  $f_n \geq f_{n+1}$  a.e. for all  $n$  such that  $\lim_{n \rightarrow \infty} f_n = 0$  a.e., but  $\lim_{n \rightarrow \infty} \int f_n \neq 0$ .

*Remark:* This shows that there is no direct analog of the Monotone Convergence Theorem for decreasing sequences of functions in  $L^+$ , unless some additional condition is imposed.

- (b) Use the Monotone Convergence Theorem (and Corollary 2.17) to show that, if  $\{f_n\}$  is a sequence in  $L^+$  such that  $f_n \geq f_{n+1}$  a.e. for all  $n$ ,  $\int f_1 < \infty$ , and  $f := \lim_{n \rightarrow \infty} f_n (= \inf_n f_n)$ , then  $\int f = \lim_{n \rightarrow \infty} \int f_n$ .

**Additional problem 2.** In all parts of this problem,  $m$  stands for the Lebesgue measure.

- (a) Let  $f_n := \chi_{(n, n+1)} : \mathbb{R} \rightarrow \mathbb{R}$ . Show that  $f_n \rightarrow 0$  pointwise and find  $\int f_n dm$ . Does your result contradict the Monotone Convergence Theorem?
- (b) Let  $f_n := n\chi_{(0, 1/n)} : [0, 1] \rightarrow \mathbb{R}$ . Find the pointwise limit of this function sequence and compute  $\int f_n dm$ . Explain why this example does not contradict the Monotone Convergence Theorem.
- (c) Construct an example of a sequence of measurable functions  $\{f_n\}$  defined on  $[0, 1]$  and taking values in  $[0, \infty]$ , such that

$$\int \liminf f_n dm < \liminf \int f_n dm .$$

**Additional problem 3.**

- (a) Let  $\epsilon$  be an arbitrary number in  $(0, 1)$ . Carry out the construction of the generalized Cantor set (see page 30 of Folland's book) to construct such a set with Lebesgue measure at least  $1 - \epsilon$ .
- (b) Use your result from part (a) to prove that for every  $\epsilon \in (0, 1)$  there exists an open dense subset  $V \subset \mathbb{R}$  whose Lebesgue measure does not exceed  $\epsilon$ .

**Additional problem 4.** Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ , and

$$E := \left\{ x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) \text{ exists in } \mathbb{R} \right\} .$$

(note that we might have  $E = \emptyset$  because we do not make any specific assumption about the pointwise convergence of  $\{f_n\}_{n=1}^{\infty}$ ). Prove that  $E$  is a countable intersection of countable unions of closed sets.

*Hint:* Recall that, by definition, a sequence  $\{a_n\}_{n=1}^{\infty}$  of real numbers is Cauchy if for each  $\epsilon$  there exists a number  $N$  such that  $m, n > N$  implies that  $|a_m - a_n| < \epsilon$  (or, equivalently, if  $m, n > N$  implies  $|a_m - a_n| \leq \epsilon$ ). Use that for  $x \in \mathbb{R}$ , the sequence of real numbers  $\{f_n(x)\}_{n=1}^{\infty}$  converges if and only if it is Cauchy. Try to write the statement “ $\{f_n(x)\}_{n=1}^{\infty}$  is Cauchy” as an equivalent statement involving the membership of  $x$  in some set-theoretic combination of the sets

$$E_{ijk} = \left\{ y \in \mathbb{R} : |f_i(y) - f_j(y)| \leq \frac{1}{k} \right\} ,$$

where  $i, j$  and  $k$  are natural numbers. Define the family of functions  $\{g_{ij}\}_{i,j=1}^{\infty}$  by  $g_{ij}(x) := |f_i(x) - f_j(x)|$ . Are these functions continuous? How can you write the sets  $E_{ijk}$  in terms of the functions  $g_{ij}$ ? What do you know about the pre-images of closed sets under a continuous function? Finally, what do you know about intersection of closed sets?