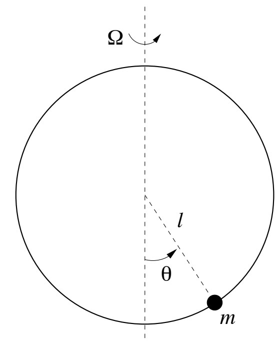
MATH 4193/5103 Homework 6 Due 5/4/17 (Thursday)

Problem 1. [A bead on a rotating hoop]

A bead of mass m can slide without friction on a circular hoop of radius ℓ that rotates about a vertical diameter with constant angular speed Ω as shown in the figure.



The equation of motion of the bead can be shown to be

$$m\ell \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = m\ell\,\Omega^2 \cos\theta \sin\theta - mg\sin\theta \ , \tag{1}$$

where the angle θ belongs to the circle S^1, which is nothing but the the interval $(-\pi, \pi]$ with identified ends (if you are more versed in mathematics, you can write $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$). By introducing the dimensionless time $\tau := t\sqrt{\frac{g}{\ell}}$ and the non-negative dimensionless parameter $\mu := \frac{\ell\Omega}{g} \geq 0$, we can rewrite (1) as the system

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \nu \ , \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\tau} = (\mu\cos\theta - 1)\sin\theta \ . \tag{2}$$

The parameter μ is the square of the ratio of the angular velocity Ω of the hoop's rotation and the frequency $\sqrt{\frac{g}{\ell}}$ of the small oscillations of the bead when the hoop is not rotating.

(a) Find all fixed points (i.e., equilibrium solutions) of the system (2). Show that, if $\mu \leq 1$, there are two equilibria, while for $\mu > 1$ there are four equilibria.

(b) Linearize (2) around the fixed point $(\pi, 0)$. What kind of fixed point is it? Is it hyperbolic?

Hint: If (2) is written as $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x} = \mathbf{f}(\mathbf{x})$, then $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \mu(\cos^2\theta - \sin^2\theta) - \cos\theta & 0 \end{pmatrix}$.

(c) In the case $\mu < 1$, linearize (2) around the fixed point $(0, 0)$, and show that $(0, 0)$ is a center (hence, non-hyperbolic). Find the period of the small periodic motion around this fixed point as a function of the parameter μ.

Hint: If $\lambda_{1,2}$ are the eigenvalues of the matrix of the linearized system (recall that λ_1 is the complex conjugate of λ_2), then in the case of a center the period of the small periodic motions around the corresponding fixed point is $\frac{2\pi}{\mathrm{Im}\,\lambda}$.

(d) In the case $\mu > 1$, linearize (2) around the fixed point $(0, 0)$. What kind of fixed point is $(0, 0)$ in this case? Is it hyperbolic? Find its eigenvalues and eigenvectors.

(e) In the case $\mu > 1$, linearize (2) around the fixed point $(\arccos \frac{1}{\mu}, 0)$ and show that it is a center. Find the period of the small periodic motion around this fixed point as a function of the parameter μ.

(f) Sketch the position of the four equilibria as functions of μ (use solid line for the stable equilibria and dashed line for the unstable ones). Find the positions of the four equilibria in the limit $\mu \to \infty$. What is the physical explanation of your result (in particular, in the limit $\mu \to \infty$)?

(g) What is the physical explanation of the bifurcation occurring at $\mu = 1$?

(h) **Only if you take the class as 5103!**

Use your results from (d) and (e) to sketch the phase portrait of the system in the case $\mu > 1$.

Remark: The behavior of the system around the fourth fixed point, $(-\arccos \frac{1}{\mu}, 0)$, is the same as around $(\arccos \frac{1}{\mu}, 0)$.

(i) **Only if you take the class as 5103!**

Let $\mu(\Omega)$ be the frequency of the small oscillations of the bead around the stable equilibrium solutions as a function of the rotation frequency Ω. Plot $\mu(\Omega)$ for $\Omega \in [0, 3\omega_0]$. Show that $\mu(\Omega)$ has a singularity of a cusp type at $\Omega = \omega_0$ (i.e., that $\lim_{\Omega\to\omega_0-} \mu(\omega) = -\infty$ and $\lim_{\Omega\to\omega_0+} \mu(\omega) = \infty$). What does this imply for the period, $T(\Omega) := \frac{2\pi}{\mu(\Omega)}$?

Problem 2. ["Traveling front" solutions of a nonlinear PDE]

In this problem you will find the allowed range of solutions of a nonlinear equation. Consider the equation

$$\frac{\partial \widetilde{u}}{\partial \widetilde{t}} + \varepsilon \frac{\partial \widetilde{u}}{\partial \widetilde{x}} = D \frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2} + r\widetilde{u}\left(1 - \frac{\widetilde{u}}{K}\right) , \qquad x \in \mathbb{R} , \quad t > 0 . \tag{3}$$

Here ε, D, r, and K are positive constants. The solution $\widetilde{u}$ can be interpreted as population or concentration, so we require that it is *positive* for any x and t.

(a) If $\widetilde{u}$ is measured in kg (kilograms), $\widetilde{x}$ is measured in m (meters), and $\widetilde{t}$ is measured in s (seconds), what are the units of ε, D, r, and K?

Hint: You can reason like this: the unit of $\frac{\partial \widetilde{u}}{\partial \widetilde{t}}$ is $\left[\frac{\partial \widetilde{u}}{\partial \widetilde{t}}\right] = \frac{[\widetilde{u}]}{[\widetilde{t}]} = \frac{\text{kg}}{\text{s}}$; similarly, the unit for measuring, say, $D\frac{\partial^2 \widetilde{u}}{\partial \widetilde{x}^2}$ is $[D]\,\frac{\text{kg}}{\text{m}^2}$; these two units must be equal, so $\frac{\text{kg}}{\text{s}} = [D]\,\frac{\text{kg}}{\text{m}^2}$, therefore the unit of D is $[D] = \frac{\text{m}^2}{\text{s}}$.

(b) Define new quantities, x, t, and u, as follows:

$$\widetilde{u} = Ku \ , \qquad \widetilde{T} = \frac{t}{r} \ , \qquad \widetilde{x} = \sqrt{\frac{D}{r}}\, x \ ,$$

and show that the PDE (3) becomes

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2} + u\,(1-u) \ , \qquad \mu = \text{const} > 0 \ . \tag{4}$$

How is the new positive constant μ related to the original constants ε, D, r, and K?

(c) Look for solutions of (4) that represent a traveling front with constant profile, i.e.,

$$u(x,t) = U(x-ct) \ ,$$

where $c = \text{const} > 0$ is a positive constant (the speed of the front), and U is a function of one variable satisfying the conditions

$$\lim_{z\to-\infty} U(z) = \text{const} \ , \qquad \lim_{z\to\infty} U(z) = 0 \ , \qquad U(z) \geq 0 \ \text{ for all } \ z \in \mathbb{R} \ . \tag{5}$$

Start by expressing the derivatives $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial x}$, and $\dfrac{\partial^2 u}{\partial x^2}$, in terms of $U'(z)$ and $U''(z)$ (where $z = x - ct$).

(d) Using your results from part (c), rewrite the PDE (4) for $u(x,t)$ as a second order ODE for $U(z)$.

(e) Define the new function $V(z) := U'(z)$, and rewrite the second order ODE for $U(z)$ as the following system of two first order ODEs for the functions $U(z)$ and $V(z)$:

$$\begin{aligned} U'(z) &= V \ , \\ V'(z) &= -U(1-U) - (c-\mu)\,V \ . \end{aligned} \tag{6}$$

(f) The system (6) has two fixed points: $(0,0)$ and $(1,0)$. Show that the linearization of the system at the point $(1,0)$ is $\begin{pmatrix} 0 & 1 \\ 1 & -(c-\mu) \end{pmatrix}$, find its eigenvalues and explain why the fixed point $(1,0)$ is always a saddle.

(g) Show that the linearization of the system at the point $(0,0)$ is $\begin{pmatrix} 0 & 1 \\ -1 & -(c-\mu) \end{pmatrix}$, and write the characteristic equation for the eigenvalues λ.

(h) As we discussed in class, the condition that $U(z)$ be positive (recall (5)) is violated if the fixed point $(0,0)$ is a spiral (because then the trajectory of the system (6) in the (U,V)-plane will enter the region $\{U < 0\}$). Find the condition on the speed c for *non-existence* of "traveling front" solutions of the PDE (4). How does the "critical" speed c (below which there are no "traveling front" solutions) depend on the parameter μ?