MATH 4193/5103 HW 6 Due by 11:59 pm on 5/2 (Sat)

Food for Thought Problem 1.[1] **[Thinking simply, again: self-similarity]**

In Problem 1 of Homework 4 you studied the *golden mean* γ, which is defined as the expression

$$\gamma = \cfrac{1}{1+\cfrac{\mathbf{1}}{\mathbf{1}+\cfrac{\mathbf{1}}{\mathbf{1}+\cfrac{\mathbf{1}}{\mathbf{1}+\cdots}}}} \,. \tag{1}$$

There you obtained the golden mean as a result of an iterative procedure, and proved rigorously the existence and uniqueness of the fixed point of this procedure.

(a) In Food for Thought Problem 1 of Homework 5 you found the resistance of an infinite system of resistors by noticing the self-similarity of the system. Can you use the same idea to find simply the value of the golden mean?

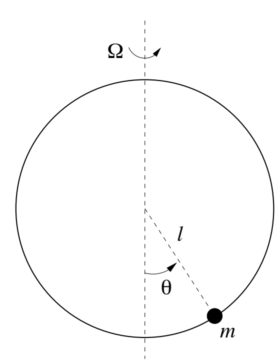
Hint: Look at the part of the right-hand side of (1) written with bold face digits. How it is related to γ?

(b) Can you apply the same idea to find the value of the number

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{\mathbf{1}}{\mathbf{1}+\cfrac{\mathbf{1}}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{1}+\cfrac{\mathbf{1}}{\mathbf{2}+\cdots}}}}}} \;?$$

Problem 2. [A bead on a rotating hoop]

A bead of mass m can slide without friction on a circular hoop of radius ℓ that rotates about a vertical diameter with constant angular speed Ω as shown in the figure.



[1] **Food for Thought problems are NOT to be turned it, they are just for fun.**

The equation of motion of the bead can be shown to be

$$m\ell \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = m\ell\,\Omega^2 \cos\theta \sin\theta - mg\sin\theta \ , \tag{2}$$

where the angle θ belongs to the circle S^1, which is nothing but the the interval $(-\pi, \pi]$ with identified ends (if you are more versed in mathematics, you can write $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$). By introducing the dimensionless time $\tau := t\sqrt{\frac{g}{\ell}}$ and the non-negative dimensionless parameter $\mu := \frac{\ell\Omega}{g} \geq 0$, we can rewrite (2) as the system

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \nu \ , \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\tau} = (\mu\cos\theta - 1)\sin\theta \ . \tag{3}$$

The parameter μ is the square of the ratio of the angular velocity Ω of the hoop's rotation and the frequency $\sqrt{\frac{g}{\ell}}$ of the small oscillations of the bead when the hoop is not rotating.

(a) Find all fixed points (i.e., equilibrium solutions) of the system (3). Show that, if $\mu \leq 1$, there are two equilibria, while for $\mu > 1$ there are four equilibria.

(b) Linearize (3) around the fixed point $(\pi, 0)$. What kind of fixed point is it? Is it hyperbolic?

Hint: If (3) is written as $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x} = \mathbf{f}(\mathbf{x})$, then $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \mu(\cos^2\theta - \sin^2\theta) - \cos\theta & 0 \end{pmatrix}$.

(c) In the case $\mu < 1$, linearize (3) around the fixed point $(0, 0)$, and show that $(0, 0)$ is a center (hence, non-hyperbolic). Find the period of the small periodic motion around this fixed point as a function of the parameter μ.

Hint: If $\lambda_{1,2}$ are the eigenvalues of the matrix of the linearized system (recall that λ_2 is the complex conjugate of λ_1), then in the case of a center the period of the small periodic motions around the corresponding fixed point is $\frac{2\pi}{|\mathrm{Im}\,\lambda_1|}$.

(d) In the case $\mu > 1$, linearize (3) around the fixed point $(0, 0)$. What kind of fixed point is $(0, 0)$ in this case? Is it hyperbolic? Find its eigenvalues and eigenvectors.

(e) In the case $\mu > 1$, linearize (3) around the fixed point $(\arccos\frac{1}{\mu}, 0)$ and show that it is a center. Find the period of the small periodic motion around this fixed point as a function of the parameter μ.

(f) Sketch the position of the four equilibria as functions of μ (use solid line for the stable equilibria and dashed line for the unstable ones). Find the positions of the four equilibria in the limit $\mu \to \infty$. What is the physical explanation of your result (in particular, in the limit $\mu \to \infty$)?

(g) What is the physical explanation of the bifurcation occurring at $\mu = 1$?

(h) **Only if you take the class as 5103!**

Use your results from (d) and (e) to sketch the phase portrait of the system in the case $\mu > 1$.

Remark: The behavior of the system around the fourth fixed point, $(-\arccos\frac{1}{\mu}, 0)$, is the same as around $(\arccos\frac{1}{\mu}, 0)$.

(i) **Only if you take the class as 5103!**

Let $\eta(\Omega)$ be the frequency of the small oscillations of the bead around the stable equilibrium solutions as a function of the rotation frequency Ω. Plot $\eta(\Omega)$ for $\Omega \in [0, 3\omega_0]$. Show that $\eta(\Omega)$ has a singularity of a cusp type at $\Omega = \omega_0$ (i.e., that $\lim\limits_{\Omega\to\omega_0-} \eta(\omega) = -\infty$ and $\lim\limits_{\Omega\to\omega_0+} \eta(\omega) = \infty$). What does this imply for the period, $T(\Omega) := \frac{2\pi}{\eta(\Omega)}$?

Problem 3. [Liénard plane; Lyapunov function for the van der Pol equation]
Let f and g be functions of one variable. Consider the equation

$$x'' + f(x)x' + g(x) = 0 \ . \tag{4}$$

(a) Show that (4) can be rewritten as a system of first-order equations as follows:

$$\begin{aligned} x' &= y - F(x) \ , \\ y' &= -g(x) \ , \end{aligned} \tag{5}$$

where $F(x) := \displaystyle\int_0^x f(s)\,\mathrm{d}s$. Now everything is happening in the (x, y) plane, which for the particular choice (5) is called the *Liénard plane.*

Hint: Set $y := x' + F(x)$, use the Fundamental Theorem of Calculus and the Chain Rule.

(b) Write down (5) for the particular choice of the van der Pol's equation,

$$x'' + \mu(1 - x^2)x' + x = 0 \ , \qquad \mu = \text{const} > 0 \ . \tag{6}$$

(c) Use the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as Lyapunov function for the system of first-order ODEs that you wrote in part (b) (corresponding to the van der Pol's equation (6)) to show that there are no periodic orbits of that system in the region $\{-\sqrt{3} < x < \sqrt{3}\}$ $(x \neq 0)$.

Hint: Compute $\frac{\mathrm{d}}{\mathrm{d}t}V(x(t), y(t))$ by using the Chain Rule and the expressions for x' and y' from (5). Look at the sign of this derivative in the region $\{-\sqrt{3} < x < \sqrt{3}\}$ $(x \neq 0)$. What can you conclude about the behavior of $V(x(t), y(t))$ with time and what does this imply about the existence of periodic orbits?