

Food for Thought Problem 1.¹ [Thinking simply, again: self-similarity]

In Problem 1 of Homework 4 you studied the *golden mean* γ , which is defined as the expression

$$\gamma = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}} . \quad (1)$$

There you obtained the golden mean as a result of an iterative procedure, and proved rigorously the existence and uniqueness of the fixed point of this procedure.

- (a) In Food for Thought Problem 1 of Homework 5 you found the resistance of an infinite system of resistors by noticing the self-similarity of the system. Can you use the same idea to find simply the value of the golden mean?

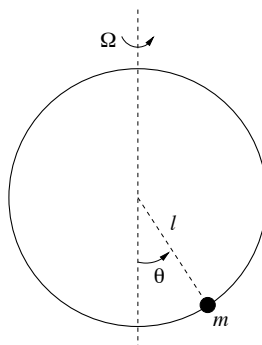
Hint: Look at the part of the right-hand side of (1) written with bold face digits. How it is related to γ ?

- (b) Can you apply the same idea to find the value of the number

$$\frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \dots}}}}}} ?$$

Problem 2. [A bead on a rotating hoop]

A bead of mass m can slide without friction on a circular hoop of radius ℓ that rotates about a vertical diameter with constant angular speed Ω as shown in the figure.



¹Food for Thought problems are NOT to be turned in, they are just for fun.

The equation of motion of the bead can be shown to be

$$m\ell \frac{d^2\theta}{dt^2} = m\ell \Omega^2 \cos\theta \sin\theta - mg \sin\theta , \quad (2)$$

where the angle θ belongs to the circle S^1 , which is nothing but the interval $(-\pi, \pi]$ with identified ends (if you are more versed in mathematics, you can write $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$). By introducing the dimensionless time $\tau := t\sqrt{\frac{g}{\ell}}$ and the non-negative dimensionless parameter $\mu := \frac{\ell\Omega}{g} \geq 0$, we can rewrite (2) as the system

$$\frac{d\theta}{d\tau} = \nu , \quad \frac{d\nu}{d\tau} = (\mu \cos\theta - 1) \sin\theta . \quad (3)$$

The parameter μ is the square of the ratio of the angular velocity Ω of the hoop's rotation and the frequency $\sqrt{\frac{g}{\ell}}$ of the small oscillations of the bead when the hoop is not rotating.

- (a) Find all fixed points (i.e., equilibrium solutions) of the system (3). Show that, if $\mu \leq 1$, there are two equilibria, while for $\mu > 1$ there are four equilibria.
- (b) Linearize (3) around the fixed point $(\pi, 0)$. What kind of fixed point is it? Is it hyperbolic?

Hint: If (3) is written as $\frac{d}{d\tau}\mathbf{x} = \mathbf{f}(\mathbf{x})$, then $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \mu(\cos^2\theta - \sin^2\theta) - \cos\theta & 0 \end{pmatrix}$.

- (c) In the case $\mu < 1$, linearize (3) around the fixed point $(0, 0)$, and show that $(0, 0)$ is a center (hence, non-hyperbolic). Find the period of the small periodic motion around this fixed point as a function of the parameter μ .

Hint: If $\lambda_{1,2}$ are the eigenvalues of the matrix of the linearized system (recall that λ_2 is the complex conjugate of λ_1), then in the case of a center the period of the small periodic motions around the corresponding fixed point is $\frac{2\pi}{|\operatorname{Im} \lambda_1|}$.

- (d) In the case $\mu > 1$, linearize (3) around the fixed point $(0, 0)$. What kind of fixed point is $(0, 0)$ in this case? Is it hyperbolic? Find its eigenvalues and eigenvectors.
- (e) In the case $\mu > 1$, linearize (3) around the fixed point $(\arccos \frac{1}{\mu}, 0)$ and show that it is a center. Find the period of the small periodic motion around this fixed point as a function of the parameter μ .
- (f) Sketch the position of the four equilibria as functions of μ (use solid line for the stable equilibria and dashed line for the unstable ones). Find the positions of the four equilibria in the limit $\mu \rightarrow \infty$. What is the physical explanation of your result (in particular, in the limit $\mu \rightarrow \infty$)?
- (g) What is the physical explanation of the bifurcation occurring at $\mu = 1$?

- (h) **Only if you take the class as 5103!**

Use your results from (d) and (e) to sketch the phase portrait of the system in the case $\mu > 1$.

Remark: The behavior of the system around the fourth fixed point, $(-\arccos \frac{1}{\mu}, 0)$, is the same as around $(\arccos \frac{1}{\mu}, 0)$.

- (i) **Only if you take the class as 5103!**

Let $\eta(\Omega)$ be the frequency of the small oscillations of the bead around the stable equilibrium solutions as a function of the rotation frequency Ω . Plot $\eta(\Omega)$ for $\Omega \in [0, 3\omega_0]$. Show that $\eta(\Omega)$ has a singularity of a cusp type at $\Omega = \omega_0$ (i.e., that $\lim_{\Omega \rightarrow \omega_0^-} \eta(\omega) = -\infty$ and $\lim_{\Omega \rightarrow \omega_0^+} \eta(\omega) = \infty$). What does this imply for the period, $T(\Omega) := \frac{2\pi}{\eta(\Omega)}$?

Problem 3. [Liénard plane; Lyapunov function for the van der Pol equation]

Let f and g be functions of one variable. Consider the equation

$$x'' + f(x)x' + g(x) = 0 . \quad (4)$$

- (a) Show that (4) can be rewritten as a system of first-order equations as follows:

$$\begin{aligned} x' &= y - F(x) , \\ y' &= -g(x) , \end{aligned} \quad (5)$$

where $F(x) := \int_0^x f(s) ds$. Now everything is happening in the (x, y) plane, which for the particular choice (5) is called the *Liénard plane*.

Hint: Set $y := x' + F(x)$, use the Fundamental Theorem of Calculus and the Chain Rule.

- (b) Write down (5) for the particular choice of the van der Pol's equation,

$$x'' + \mu(1 - x^2)x' + x = 0 , \quad \mu = \text{const} > 0 . \quad (6)$$

- (c) Use the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ as Lyapunov function for the system of first-order ODEs that you wrote in part (b) (corresponding to the van der Pol's equation (6)) to show that there are no periodic orbits of that system in the region $\{-\sqrt{3} < x < \sqrt{3}\}$ ($x \neq 0$).

Hint: Compute $\frac{d}{dt}V(x(t), y(t))$ by using the Chain Rule and the expressions for x' and y' from (5). Look at the sign of this derivative in the region $\{-\sqrt{3} < x < \sqrt{3}\}$ ($x \neq 0$). What can you conclude about the behavior of $V(x(t), y(t))$ with time and what does this imply about the existence of periodic orbits?