MATH 4163 **Homework 6** **Due Thu, 03/08/2018**

Problem 1. [Vector calculus and integral indentities]

(a) Let $u : \mathbb{R}^3 \to \mathbb{R}$ be a scalar function on $\mathbb{R}^3$, and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector field on $\mathbb{R}^3$. Recall that, if one uses Cartesian coordinates (x, y, z), then the dot product of two vectors, $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, is $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, and the gradient of a scalar function and the divergence of a vector field on $\mathbb{R}^3$ are defined as

$$\text{grad} \equiv \nabla u := \frac{\partial u}{\partial x}\,\mathbf{i} + \frac{\partial u}{\partial y}\,\mathbf{j} + \frac{\partial u}{\partial z}\,\mathbf{k}\,, \qquad \operatorname{div}\mathbf{F} \equiv \nabla \cdot \mathbf{F} := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,,$$

where $\mathbf{F}(\mathbf{x}) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$. Prove the identity

$$\nabla \cdot (u\mathbf{F}) = (\nabla u) \cdot \mathbf{F} + u\, \nabla \cdot \mathbf{F}\,.$$

Which differentiation rule have you used in your derivation?

(b) Let D be a domain in $\mathbb{R}^3$ with boundary ∂D with outward normal vector $\mathbf{n}$, $\mathrm{d}S$ be the area element on ∂D, and $\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S$. Let $u : \mathbb{R}^3 \to \mathbb{R}$ and $v : \mathbb{R}^3 \to \mathbb{R}$ be scalar functions on $\mathbb{R}^3$, and $\Delta = \nabla \cdot \nabla$ be the Laplacian acting on scalar functions. Apply the Divergence Theorem from Calculus to your result from part (a) to prove the identity

$$\iiint_D u\, \Delta v \,\mathrm{d}V = -\iiint_D (\nabla u) \cdot (\nabla v)\,\mathrm{d}V + \oint\!\!\!\oint_{\partial D} u\, \nabla v \cdot \mathrm{d}\mathbf{S}\,.$$

(c) Using the notations from part (b), prove the identity

$$\iiint_D (u\, \Delta v - v\, \Delta u)\,\mathrm{d}V = \oint\!\!\!\oint_{\partial D} (u\, \nabla v - v\, \nabla u) \cdot \mathrm{d}\mathbf{S}\,.$$

Remark: Analogous results hold for scalar functions $u : \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}$ and vector fields $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, where gradient, div, and Δ (acting on scalar functions) are defined by

$$\nabla u(\mathbf{x}) := \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(\mathbf{x})\,\mathbf{e}_i\,, \quad \nabla \cdot \mathbf{F}(\mathbf{x}) := \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}(\mathbf{x})\,, \quad \Delta u(\mathbf{x}) := \nabla \cdot \nabla u(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(\mathbf{x})\,,$$

$(x_1, \ldots, x_n)$ are the Cartesian coordinates in $\mathbb{R}^n$ and $\mathbf{e}_i$ is the unit vector in direction of x_i.

Problem 2. [Spaces of infinite sequences]

Many theorems that hold in finite-dimensional spaces are not true in infinite-dimensional spaces. One can think of the real infinite-dimensional space $\mathbb{R}^\infty$ as the space of infinite sequences: $\mathbf{u} = (u_1, u_2, u_3, \ldots)$, where u_j are real numbers ($j \in \mathbb{N} := \{1, 2, 3, \ldots\}$). In this space we can define the norms $\| \ \|_1$, $\| \ \|_2$, and $\| \ \|_\infty$ as usual:

$$\|\mathbf{u}\|_1 := \sum_{j\in\mathbb{N}} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j\in\mathbb{N}} |u_j|^2 \right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \sup_{j\in\mathbb{N}} |u_j| \ .$$

Here $\sup_{j\in\mathbb{N}} a_j$ (the "supremum") is the smallest number a such that $a_j \leq a$ for all $j \in \mathbb{N}$. The supremum over a finite set of real numbers is the same as the maximum over this set. For an infinite set, however, the set may not have a maximum, but it always has a supremum (which may be finite or infinite); for example, the set $\{5-1, 5-\frac{1}{2}, 5-\frac{1}{3}, 5-\frac{1}{4}, \ldots, 5-\frac{1}{k}, \ldots\}$ has a supremum (equal to 5), but does not have a maximum (because none of the elements of the set is *equal* to 5).

The notations ℓ^1, ℓ^2, and ℓ^∞ are sometimes used for the spaces of infinite sequences whose $\|\mathbf{u}\|_1$, $\|\mathbf{u}\|_2$, or $\|\mathbf{u}\|_\infty$, are finite:

$$\begin{aligned} \ell^1 &:= \{\mathbf{u} \in \mathbb{R}^\infty \ : \ \|\mathbf{u}\|_1 < \infty\} \ , \\ \ell^2 &:= \{\mathbf{u} \in \mathbb{R}^\infty \ : \ \|\mathbf{u}\|_2 < \infty\} \ , \\ \ell^\infty &:= \{\mathbf{u} \in \mathbb{R}^\infty \ : \ \|\mathbf{u}\|_\infty < \infty\} \ . \end{aligned}$$

One can show that $\ell^1 \subseteq \ell^2 \subseteq \ell^\infty$. In this problem you will give examples showing that these inclusions are strict, i.e., that there exist vectors that are in ℓ^2 but not in ℓ^1, and there exist vectors that are in ℓ^∞ but not in ℓ^2.

(a) Show that the sequence $\mathbf{u} = (1, 1, 1, \ldots)$ belongs to ℓ^∞, but not to ℓ^1.

(b) Give an explicit example of a sequence $\mathbf{v}$ such that $\|\mathbf{v}\|_\infty < \infty$, but $\|\mathbf{v}\|_2$ is infinite.

(c) Give an explicit example of a sequence $\mathbf{w}$ such that $\|\mathbf{w}\|_2 < \infty$, but $\|\mathbf{w}\|_1$ is infinite.
Hint: Think how you can use the following facts:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \ , \qquad \sum_{j=1}^{\infty} \frac{1}{j} = \infty \ .$$

Problem 3. [Transforming an eigenvalue problem into a Sturm-Liouville form]
The *Sturm-Liouville form* of an eigenvalue problem is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[K(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)y(x) + \lambda g(x)y(x) = 0 \ . \tag{1}$$

Any eigenvalue problem of the form

$$\alpha(x)y''(x) + \beta(x)y'(x) + \gamma(x)y(x) + \lambda\delta(x)y(x) = 0 \ , \tag{2}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $\delta(x)$ are given functions, can be written in Sturm-Liouville form. To this end, one has to multiply (2) by an appropriately chosen *integrating factor* $m(x)$, i.e., a function $m(x)$ such that the resulting equation has the form (1). One can show that the integrating factor can be chosen to be

$$m(x) = \frac{1}{\alpha(x)} \exp\left\{ \int \frac{\beta(x)}{\alpha(x)} \, \mathrm{d}x \right\} . \tag{3}$$

(a) Consider the eigenvalue problem consisting of the ODE

$$(1 - x^2) y''(x) - x y'(x) + \lambda y(x) = 0 , \qquad x \in [-1, 1] \tag{4}$$

and appropriately chosen boundary conditions (which we will not specify here). Use (3) to show that the integrating factor for the problem (4) is $m(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(b) Multiply (4) by $m(x)$ from part (b) and write the resulting equation in the form (1), i.e., identify the functions $K(x)$, $q(x)$, and $g(x)$ from (1).

(c) Let the functions $T_n(x)$, $n = 0, 1, 2, 3, \ldots$ (where $x \in [-1, 1]$) be the solutions of the eigenvalue problem from part (a). What does the general theory say about the orthogonality properties of these functions? Write explicitly the inner product with respect to which the functions T_n are orthogonal.

Hint: See part (c) of the Theorem in the Appendix to this homework.

Remark: One can show that the functions $T_n(x) = \cos(n \arccos x)$ satisfy (4), and the corresponding eigenvalues is $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$; the functions $T_n(x)$ are polynomials called *Chebyshev polynomials*.

Problem 4. [A Sturm-Liouville problem]

Consider the following Sturm-Liouville eigenvalue problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^x \frac{\mathrm{d}y}{\mathrm{d}x} \right] + \frac{1}{4} \mathrm{e}^x y(x) + \lambda \mathrm{e}^x y(x) = 0 , \qquad x \in [0, \pi] , \\ &y(0) = 0 , \quad y(\pi) = 0 . \end{aligned} \tag{5}$$

(a) Check that the functions

$$y_n(x) = \mathrm{e}^{-x/2} \sin nx \tag{6}$$

are eigenfunctions of the Sturm-Liouville eigenvalue problem (5) and find the corresponding eigenvalues λ_n.

(b) Show by direct calculation that the eigenfunctions (y_m) are orthogonal with respect to the corresponding inner product.

Hint: See part (c) of the Theorem in the Appendix to this homework.

(c) Find the coefficients of the expansion of the function $f(x) = \sin(5x)$, $x \in (0, \pi)$ as a linear combination of $y_n(x)$ (6). You may use that, for $m, n \in \mathbb{Z}$, $a \in \mathbb{R}$,

$$\int \sin(mx)\,\sin(nx)\,\mathrm{e}^{ax}\,\mathrm{d}x = \frac{2amn[(-1)^{m+n}\mathrm{e}^{a\pi} - 1]}{[a^2 + (m+n)^2][a^2 + (m-n)^2]} \ .$$

Appendix: Summary of Sturm-Liouville theory (after Bleecker and Csordas)

Here we give a summary of Sturm-Liouville theory, following Section 4.4 of the book by Bleecker and Csordas.

Definition. A *Sturm-Liouville problem* (SL problem) is a boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[K(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + q(x)\,y(x) + \lambda\,g(x)\,y(x) = 0 \ , \qquad x \in [a, b] \ , \tag{7}$$

$$c_1 y(a) + c_2 y'(a) = 0 \ , \quad c_2 y(b) + c_4 y'(b) = 0 \tag{8}$$

for the function $y \in C^2([a, b])$ (i.e., y'' exists and is continuous) where K, q, and g are real-valued functions defined on $[a, b]$, and c_j $(j = 1, \ldots, 4)$ are real constants satisfying the following properties:

(i) $g(x) > 0$ on $[a, b]$, $K(x) \geq 0$ and $K(x)$ may vanish at most at a finite number of points in $[a, b]$;

(ii) $q(x)$, $g(x)$, and $K'(x)$ are continuous on $[a, b]$;

(iii) $c_1^2 + c_2^2 \neq 0$, and $c_3^2 + c_4^2 \neq 0$ (i.e., $(c_1, c_2) \neq (0, 0)$ and $(c_3, c_4) \neq (0, 0)$).

The PDE (7) is called a *Sturm-Liouville equation.* A value of the parameter λ for which a *nontrivial solution* (i.e., a solution which is not identically zero on $[a, b]$) exists is called an *eigenvalue* of the problem (7)-(8), and a corresponding nontrivial solution $y(x)$ of (7)-(8) is called an *eigenfunction* associated with that eigenvalue.

Theorem.

(a) **[Properties of eigenvalues]**

The SL problem (7)-(8) has an infinite number of real eigenvalues $\lambda_n \in \mathbb{R}$. The eigenvalues λ_n can be written in increasing order as

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \ ,$$

such that

$$\lim_{n\to\infty} \lambda_n = \infty \ .$$

(See Theorem 6 on page 267 and Theorem 9 on page 270.)

(b) **[Uniqueness of eigenfunctions = Simplicity of eigenvalues]**

If $y(x)$ and $Y(x)$ are two eigenfunctions of the SL problem (7)-(8) corresponding to the same eigenvalue λ, then $y(x) = \beta Y(x)$, $x \in [a,b]$, for some nonzero constant β. In other words, all eigenfunctions of the SL problem (7)-(8) are simple (in the sense that, up to a multiplication by a constant, there is only one eigenfunction corresponding to each eigenvalue). (See Theorem 3 on page 265.)

(c) **[Orthogonality of eigenfunctions]**

Let λ_m and λ_n be two distinct eigenvalues of the SL problem (7)-(8). Then the corresponding eigenfunctions $y_m(x)$ and $y_n(x)$ are orthogonal on $[a,b]$ with respect to the weight function $g(x)$, i.e.,

$$\int_a^b y_m(x)\, y_n(x)\, g(x)\, \mathrm{d}x = 0 \qquad \text{when } m \neq n\ .$$

(See Theorem 5 on page 267.)

(d) **[Nonnegativity of eigenvalues]**

In the SL problem (7)-(8), suppose that $q(x) \leq 0$ for $x \in [a,b]$ and that the real constants c_j $(j = 1,2,3,4)$ satisfy

$$c_1\, c_2 \leq 0\ , \quad c_3\, c_4 \geq 0\ ,$$

then all eigenvalues of (7)-(8) are nonnegative. Moreover, if 0 is an eigenvalue, then $q(x) \equiv 0$, $c_1 = c_3 = 0$, and any eigenfunction with eigenvalue 0 must be constant. (See Theorem 7 on page 268.)