MATH 4163 Homework 6 Due at noon on Fri, Oct 5, 2012

Problem 1. Consider the heat equation in a two-dimensional rectangular region $x \in [0, a]$, $y \in [0, b]$,

$$u_t(x, y, t) = \alpha^2 \Delta u(x, y, t) \ ,$$

subject to the Dirichlet-Neumann boundary conditions

$$u(0, y, t) = 0 \ , \quad u(a, y, t) = 0 \ , \qquad u_y(x, 0, t) = 0 \ , \quad u_y(x, b, t) = 0 \ ,$$

and the initial condition

$$u(x, y, 0) = f(x, y) \ .$$

(a) In class we solved a similar problem for the wave equation in a two-dimensional region, with homogeneous Dirichlet BCs on all boundaries. There we found that the general solution of the problem was given as a superposition of functions of the form $T_{nm}(t)\, S_{nm}(x, y)$, where

$$S_{nm}(x, y) = X_n(x)\, Y_m(y) \ , \qquad X_n(x) = \sin \frac{n\pi x}{a} \ , \quad Y_m(y) = \sin \frac{m\pi x}{b} \ .$$

If $\lambda_n = \frac{n\pi}{a}$ and $\mu_n = \frac{m\pi}{b}$ are the constants coming from separation of variables (this notation is slightly different from the notation in the book), then the functions X_n and Y_m can be written as $X_n(x) = \sin \lambda_n x$ and $Y_m(y) = \sin \mu_m y$.

Try the same technique for the BVP in the present problem. What should the functions $X(x)$ and $Y(y)$ be so that they satisfy the ODEs coming from the heat equation, and the corresponding BCs? What are the constants coming from the separation of variables?

(b) Write down the ODE for the function $T_{nm}(t)$ and find its general solution.

(c) Write the solution of the BVP in this problem in the form

$$u(x, y, t) = \sum_n \sum_m T_{nm}(t)\, X_n(x)\, Y_m(y) \ ,$$

and impose the IC to find the arbitrary constants in the functions $T_{nm}(t)$. (Be careful about the values that the indices n and m are allowed to take.)

Problem 2. Find the solution of the Laplace's equation,

$$\Delta u(x, y) = 0 \ ,$$

in the semi-infinite strip $x \in [0, 1]$, $y \geq 0$, subject to the boundary conditions

$$u(0, y) = 0 \ , \qquad u_x(1, y) = -hu(1, y) \ , \quad \text{where } h > 0 \ ,$$
$$u(x, 0) = 3 \ , \qquad |u(x, y)| < \infty \ .$$

Let the numbers z_n (with $n \in \mathbb{N}$) be the solutions of the transcendental equation

$$\tan z = -\frac{z}{h} \ ,$$

in increasing order: $z_1 < z_2 < z_3 < \ldots$.

(a) Write the BVP for X_n coming from the separation of variables, and express its eigenvalues and eigenfunctions in terms of the numbers z_n.

(b) Find the functions Y_n that satisfy the ODE coming from the separation of variables, as well as the condition for boundedness of the solution.

(c) Write down the solution $u(x, y)$ of the PDE as a superposition of the functions $u_n(x, y) = X_n(x)\, Y_n(y)$; do not impose the BC at $y = 0$ yet.

(d) Use the BC at $y = 0$ to find the coefficients in the expansion of $u(x, y)$ from part (c).
Hint: This is easy if you apply the general theory.

Problem 3. Separate variables in the Laplace's equation $\Delta u = 0$ in the cylindrical domain

$$r \in [0, a] \ , \quad \theta \in [0, 2\pi) \ , \quad z \in [0, b] \ .$$

The Laplacian in cylindrical coordinates (r, θ, z) has the form

$$\Delta u(r, \theta, z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \ .$$

Assume that on the bottom wall and on the side walls the function u has value zero, i.e.,

$$u(r, \theta, 0) = 0 \ , \qquad u(a, \theta, z) = 0 \ ,$$

while on the top wall it has values given by the function f:

$$u(r, \theta, b) = f(r, \theta) \ .$$

In addition, we require that the function u be bounded.

(a) Look for solution of the problem of the form

$$u(r, \theta, z) = R(r)\, \Theta(\theta)\, Z(z) \ .$$

What ODEs should the functions R, Θ and Z satisfy?

(b) The function Θ should be 2π-periodic. What are the solutions of the ODE for Θ that are 2π-periodic? Label them as $\Theta_m(\theta)$.
Remark: We did this in class – you can just copy the results.

(c) Show that the ODE for R, after an appropriate change of variables (like the one we did in class) is the Bessel's equation,

$$x^2\phi''(x) + z\phi'(x) + (x^2 - m^2)\phi(x) = 0 \ .$$

Express R in terms of the functions of Bessel, J_m, and Neumann, Y_m.

(d) Impose the condition of boundedness and the BC at $r = a$ on the radial function,

$$R_m(r) = C_m J_m(\sqrt{\lambda} r) + D_m Y_m(\sqrt{\lambda} r)$$

to obtain an explicit expression for $R_{mn}(r)$ (the index n came from numbering the zeros of J_m). Let β_{mn} be the nth zero of the function J_m.

(e) The functions Z_{mn} satisfy the harmonic oscillator equation with appropriate coefficients. Write its solutions in terms of hyperbolic functions, and use the BC at $z = 0$ to eliminate one of the two hyperbolic functions.

(f) Finally, write the series for $u(r, \theta, z)$ and impose the BC at $z = b$ to find all the coefficients in the series. Write explicitly the expressions for the coefficients. Use the orthogonality relation

$$\int_0^a J_m\left(\frac{\beta_{mn} r}{a}\right) J_m\left(\frac{\beta_{mn'} r}{a}\right) r \, \mathrm{d}r = \frac{a^2}{2} \left[J'_m(\beta_{mn})\right]^2 \delta_{nn'} \ .$$

Problem 4. The *Sturm-Liouville form* of an eigenvalue problem is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}\right] + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0 \ . \tag{1}$$

Any eigenvalue problem of the form

$$\alpha(x)\phi''(x) + \beta(x)\phi'(x) + \gamma(x)\phi(x) + \lambda\delta(x)\phi(x) = 0 \tag{2}$$

can be written in Sturm-Liouville form. To this end, one has to multiply (2) by an appropriately chosen *integrating factor* $\mu(x)$, i.e., a function $\mu(x)$ such that the resulting equation has the form (1). One can show that the integrating factor can be chosen to be

$$\mu(x) = \frac{1}{\alpha(x)} \exp\left\{\int \frac{\beta(x)}{\alpha(x)} \, \mathrm{d}x\right\} \ . \tag{3}$$

(a) Consider the eigenvalue problem consisting of the ODE

$$(1 - x^2)\phi''(x) - x\phi'(x) + \lambda\phi(x) = 0 \ , \qquad x \in [-1, 1] \tag{4}$$

and appropriately chosen additional conditions (which we will not specify here). Use (3) to show that the integrating factor for the problem (4) is $\mu(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(b) Multiply (4) by $\mu(x)$ from part (b) and write the resulting equation in the form (1), i.e., identify the functions $p(x)$, $q(x)$, and $\sigma(x)$ from (1).

(c) Let the functions $T_n(x)$, $n = 0, 1, 2, 3, \ldots$ (where $x \in [-1, 1]$) be the solutions of the eigenvalue problem from part (a). What does the general theory say about the orthogonality properties of these functions? Write explicitly the inner product with respect to which the functions T_n are orthogonal.

Remark: One can show that the explicit expression for T_n is $T_n(x) = \cos(n \arccos x)$, and the corresponding eigenvalue is $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$.