MATH 4073 Homework 6 Due Mon, 10/24/16

Problem 1. A thin wooden rod is attached to the point with coordinates $(0, 1)$ in the (x, y)-plane, and it is clamped at this point, so that it starts off in positive x-direction. The other end of the rod passes under a thin peg at the point $(10, 0)$. The rod is shown in Figure 1. Let the function

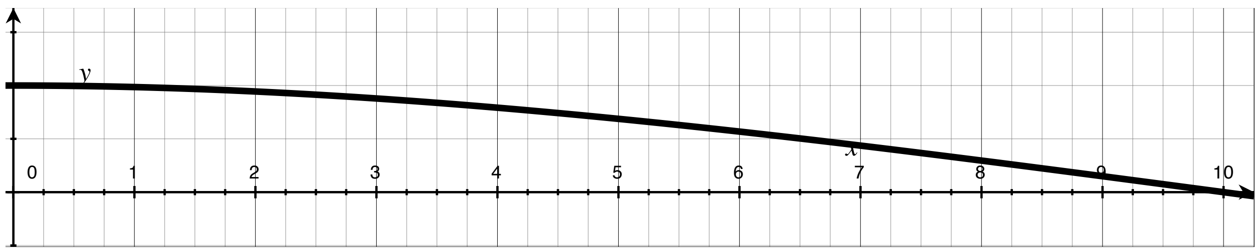

Figure 1: Shape of the wooden rod (see text).

$f(x)$ give the shape of the rod: $y = f(x)$. One can show that the bending of a rod is governed by the fourth order ordinary differential equation

$$f^{(4)}(x) = 0 \ . \tag{1}$$

According to the description above, the function $f(x)$ must satisfy the following boundary conditions:

$$\begin{aligned}
f(0) &= 1 \quad &&\text{[the rod passes through the point } (0,1)] \ , \\
f(10) &= 0 \quad &&\text{[the rod passes through the point } (10,0)] \ , \\
f'(0) &= 0 \quad &&\text{[the rod is clamped at } (0,1) \text{ and "starts off" horizontally]} \ , \\
f''(10) &= 0 \quad &&\text{[the right end of the rod is free (no bending forces)]} \ .
\end{aligned}$$

(a) Write down the general solution of the differential equation (1). Since this is a fourth order differential equation, its general solution must contain four arbitrary constants C_1, C_2, C_3, C_4.

Remark: The answer is obvious – you do not need to know how to solve differential equations in order to solve this part of the problem.

(b) Impose the boundary conditions to find the solution of the boundary value problem

$$f^{(4)}(x) = 0 \ , \quad f(0) = 1 \ , \quad f(10) = f'(0) = f''(10) = 0 \ .$$

Hint: I found that the coefficient in front of x^3 is $\frac{1}{2000}$.

(c) Why do you think I gave you this problem in the homework that is mostly on spline interpolation? If you think of the shape of the rod as given by a cubic spline, how would you classify the spline? (Words that are potentially relevant are FREE, CLAMPED.)

Problem 2. Consider the function $f(x) = \sqrt{x}$. We want to find a cubic polynomial $S(x)$ that interpolates $f(x)$ on the interval $x \in [1, 4]$ with clamped boundary conditions at both ends.

(a) Clearly, $S(x)$ must have the same values as $f(x)$ at the points $x_0 = 1$ and $x_1 = 4$. What are the clamped boundary conditions for $S'(1)$ and $S'(4)$?

(b) Write the interpolating polynomial $S(x)$ in the form

$$S(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 \ ,$$

and impose the conditions formulated in part (a) to find the coefficients of $S(x)$.
Hint: I found that $d = \frac{1}{108}$.

(c) The function $S(x)$ that you found in (a) is good not only to approximate the values of the function $f(x)$, but also to approximate the values of the integrals and some derivatives of $f(x)$. Find the numerical value of $\int_1^3 S(x)\,dx$ and compare it with the exact value, $\int_1^3 f(x)\,\mathrm{d}x$; find the absolute and the relative errors.
Hint: You may use that the exact value of the integral is $\int_1^3 \sqrt{x}\,dx = 2\sqrt{3} - \frac{2}{3} \approx 2.79743$.

(d) Find the numerical value of $S'(2)$ and compare it with the exact value, $f'(2)$; find the absolute and the relative errors.

Problem 3. A clamped cubic spline S for a function f is given by

$$S(x) \;=\; \begin{cases} S_0(x) \;=\; 1 + x + 2x^2 \ , & \text{for } 0 \le x \le 1 \ , \\ S_1(x) \;=\; a + b(x-1) + c(x-1)^2 + d(x-1)^3 \ , & \text{for } 1 \le x \le 2 \ . \end{cases}$$

The function $f(x)$ is known to satisfy the conditions

$$f'(0) = 1 \qquad \text{and} \qquad f'(2) = 0 \ .$$

Recall that the first derivatives of the clamped splines at the endpoints (in this case, at $x = 0$ and $x = 2$) are equal to the first derivatives of the original function f at these points.
Find the values of the constants a, b, c, and d in the expression for S.

Problem 4. As you know, one way to approximate a function f of one variable is to replace it by its tangent line at some point of interest, or by the "best fitting" parabola at this point (these approximations correspond to using the first- or second-order Taylor polynomial of the function f at this point). This type of approximation, however, works very well only near this point, and can be very inaccurate over an entire *interval*.

One way to approximate a function f (of one variable) on an entire interval is the following. Choose some class of functions $\mathcal{H}$, say all linear functions. Then look for a function h from this class $\mathcal{H}$ for which the "distance" between f and h is the smallest possible. The "distance" – which is usually called "error" – can be defined in many different ways. If we want to approximate f by a function $h \in \mathcal{H}$ on the interval $[a, b]$, and we want $|f(x) - h(x)|$ to be small for all $x \in [a, b]$, then an appropriate definition for the "error" would be $E_\infty := \max\limits_{x \in [a,b]} |f(x) - h(x)|$. Another choice is to minimize $E_1 := \int_a^b |f(x) - h(x)|\, dx$, but the expressions for E_∞ and E_1 cause technical difficulties if one tries to use them in practice. The most convenient for numerical purposes expression for the error is

$$E_2 := \int_a^b [f(x) - h(x)]^2 \, dx \ ,$$

which we will use below. Incidentally, the origin of the cryptic notations E_∞, E_1, and E_2 will become clear in a month or so.

In this problem you will find the best approximation of the function $f(x) = x^3$ by a linear function, $h_{\mu,\nu}(x) := \mu x + \nu$, over the interval $[0, 1]$ if the "error" is given by the integral

$$E_f(\mu, \nu) := \int_0^1 [f(x) - h_{\mu,\nu}(x)]^2 \, dx \ . \tag{2}$$

In other words, you have to choose the values of the constants μ and ν that minimize the error $E_f(\mu, \nu)$ given by (2).

Hint: Here is a useful fact: $\displaystyle\int_0^1 [x^3 - (\mu x + \nu)]^2 \, dx = \frac{1}{7} - \frac{2}{5}\,\mu + \frac{1}{3}\,\mu^2 - \frac{1}{2}\,\nu + \mu\nu + \nu^2$.