MATH 4073 Homework 6 Due 10/15/09

Problem 1. Suppose that you want to find the interpolating polynomials to the function $f(x) = x^2$ written in Newton's divided difference form.

(a) You have some data written in the table below; fill out the missing values.

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = 1$	$f[x_0] =?$			
		$f[x_0, x_1] =?$		
$x_1 = 3$	$f[x_1] =?$		$f[x_0, x_1, x_2] = 1$	
		$f[x_1, x_2] =?$		$f[x_0, x_1, x_2, x_3] =?$
$x_2 = 6$	$f[x_2] = 36$		$f[x_1, x_2, x_3] =?$	
		$f[x_2, x_3] = 13$		
$x_3 = 7$	$f[x_3] = 49$			

(b) Take any number x_4 (not necessarily integer) different from the already used values 1, 3, 6 and 7. Find the divided differences $f[x_4]$, $f[x_3, x_4]$, $f[x_2, x_3, x_4]$, $f[x_1, x_2, x_3, x_4]$, and $f[x_0, x_1, x_2, x_3, x_4]$.

(c) Do you notice some pattern in the table? How can you explain your observation?

Problem 2. Use Newton's forward-difference formula to construct interpolating polynomials for the function $f(x) = \sin(\pi x)$ based on values of $f(x)$ at equally spaced points (note that the points x_0, x_1, x_2 and x_3 below are equally spaced with $h = \frac{1}{6}$). First write the polynomial in the form

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0) \ ,$$

where $x = x_0 + sh$, i.e., $s = \frac{x - x_0}{h}$.
Fill the missing entries in the forward differences table below, and find the following:

(a) the degree-1 forward difference polynomial $P_1(x)$ based on the points $(x_0, f(x_0)) = (0, 0)$ and $(x_1, f(x_1)) = (\frac{1}{6}, \frac{1}{2})$;

(b) the degree-2 forward difference polynomial $P_2(x)$ based on the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2)) = (\frac{1}{3}, ?)$;

(c) the degree-3 forward difference polynomial $P_3(x)$ based on the points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3)) = (\frac{1}{2}, ?)$.

Use the forward differences table below, some of whose entries are given. Do *not* compute the numerical values (i.e., leave the square roots, as in the table below, and, more importantly, leave the polynomial in Newton's forward-diference form (i.e., do *not* expand it).

x_i	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order
$x_0 = 0$	$f(x_0) = 0$			
		$\Delta f(x_0) = ?$		
$x_1 = \frac{1}{6}$	$f(x_1) = \frac{1}{2}$		$\Delta^2 f(x_0) = ?$	
		$\Delta f(x_1) = ?$		$\Delta^3 f(x_0) = \dfrac{5 - 3\sqrt{3}}{2}$
$x_2 = \frac{1}{3}$	$f(x_2) = ?$		$\Delta^2 f(x_1) = \dfrac{3 - 2\sqrt{3}}{2}$	
		$\Delta f(x_2) = \dfrac{2 - \sqrt{3}}{2}$		
$x_3 = \frac{1}{2}$	$f(x_3) = ?$			

Problem 3. Let f be a polynomial of degree no more than 3, defined on some interval $[x_0, x_1]$.

(a) Show that if f is its own free spline – i.e., a cubic spline with the free (natural) boundary conditions, – then it cannot be a cubic polynomial.

Hint: Look carefully at the definition of a degree of a polynomial on page 87.

(b) What is the most general form for an interpolating polynomial of degree no more than 3 that is its own free cubic spline?

Problem 4. A thin wooden rod is attached to the point with coordinates $(0, 10)$ in the (x, y)-plane, and it is clamped at this point, so that it starts off in positive x-direction. The other end of the rod passes under a thin peg at the point $(10, 0)$. The rod is shown in Figure 1. Let the function $f(x)$ give the shape of the rod: $y = f(x)$. One can show that the

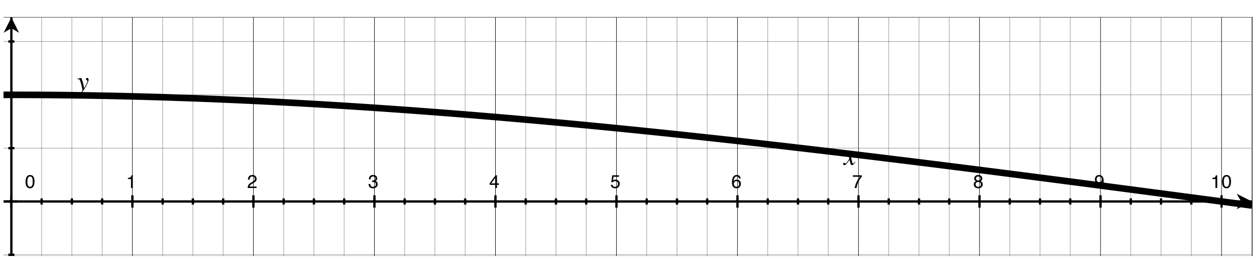

Figure 1: Shape of the wooden rod (see text).

bending of a rod is governed by the fourth order ordinary differential equation

$$f^{(4)}(x) = 0 \ . \tag{1}$$

According to the description above, the function $f(x)$ must satisfy the following boundary conditions:

$$\begin{aligned} f(0) &= 1 \quad \text{[the rod passes through the point } (0,1)] \ , \\ f(10) &= 0 \quad \text{[the rod passes through the point } (10,0)] \ , \\ f'(0) &= 0 \quad \text{[the rod is clamped at } (0,1) \text{ and “starts off” horizontally]} \ , \\ f''(10) &= 0 \quad \text{[the right end of the rod is free (no bending forces)]} \ . \end{aligned}$$

(a) Write down the general solution of the differential equation (1). Since this is a fourth order differential equation, its general solution must contain four arbitrary constants C_1, C_2, C_3, C_4.

(b) Impose the boundary conditions to find the solution of the boundary value problem

$$f^{(4)}(x) = 0 \ , \quad f(0) = 1 \ , \quad f(10) = f'(0) = f''(10) = 0 \ .$$

(c) Why do you think I gave you this problem in the homework that is mostly on spline interpolation?

Problem 5. A clamped cubic spline S for a function f is defined on $[1, 3]$ by

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3 \ , & \text{if } x \in [1,2] \ , \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 \ , & \text{if } x \in [2,3] \ . \end{cases}$$

Given that $f'(1) = f'(3)$, find the constants a, b, c, and d.