MATH 3423 Homework 6 Due Thu, 10/10/19

Problem 1.

(a) Use the formula for the sum of a geometric series, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, $z \in \mathbb{C}$, $|z| < 1$, to show that the sum of the series $\sum_{n=0}^{\infty} r^n \mathrm{e}^{\mathrm{i}n\theta}$ (where $r \geq 0$ and θ are real numbers) is

$$\sum_{n=0}^{\infty} r^n \mathrm{e}^{\mathrm{i}n\theta} = \frac{1 - r\cos\theta + \mathrm{i}\, r \sin\theta}{1 - 2r\cos\theta + r^2} \ .$$

For what values of r and θ is your result valid?

(b) Use the formula derived in part (a) to prove that $\sum_{n=0}^{\infty} r^n \cos(n\theta) = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$. For what values of r and θ is this formula valid?

(c) Use the formula for the sum of a geometric series to show that the series $\sum_{n=1}^{\infty} n r^{n-1} \mathrm{e}^{\mathrm{i}(n-1)\theta}$ (where again $r \geq 0$ and θ are real numbers) converges when $r \in [0, 1)$, and its sum is

$$\sum_{n=1}^{\infty} n r^{n-1} \mathrm{e}^{\mathrm{i}(n-1)\theta} = \frac{1}{(1 - r\mathrm{e}^{\mathrm{i}\theta})^2} \ .$$

Hint: How can you get nz^{n-1} from z^n?

Problem 2. Consider the function $f(z) = z^3 \cosh \frac{1}{z}$.

(a) Use the Taylor series $\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ to derive the Taylor series of $\cosh z$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$.

(c) The point $a = 0$ is a singularity of $f(z)$. What kind of singularity?

Problem 3. Consider the function $f(z) = \frac{1}{z^3(z-5)}$.

(a) Find and classify the singularities of $f(z)$.

(b) Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $0 < |z| < 5$.

Hint: In this and the next part of the problem use the geometric series formula.

(c) Find the Laurent expansion of $f(z)$ about the point $a = 0$ in the domain $5 < |z|$.

(d) Compute the residue of $f(z)$ at $a = 0$. If you use some of the previously obtained results to answer this question, please specify clearly what you used.

Problem 4. Find *all* Laurent expansions of $f(z) = \dfrac{1}{z-2}$ about the point $a = \mathrm{i}$.

Problem 5. Find the disk of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, z^n}{n^n}$. *Hint:* Recall the fact that $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}$.

Problem 6.

(a) Find the Taylor expansion of the function $f(z) = \dfrac{1}{1+z^2}$ about $z = 0$ by using the formula for the sum of a geometric series applied to $\dfrac{1}{1+z^2} = \dfrac{1}{1-(-z^2)}$.

(b) Use your result from (a) and the fact that $\arctan w = \displaystyle\int_0^w \frac{\mathrm{d}z}{1+z^2}$ to obtain the Taylor expansion of $\arctan w$ about $w = 0$.

Remark: It can be proven that one can integrate a convergent power series term by term, and the new series has the same disk of convergence.

Problem 7. Consider the function

$$f(z) = \frac{\mathrm{e}^{3z}}{(z-5)^4} \; .$$

(a) What is the nature of the singularity of $f(z)$ at $a = 5$?

Hint: It is easy to answer this question by using the fact stated on page 914 of the book.

(b) Find the Laurent expansion of $f(z)$ about $a = 5$.

Hint: Look at Example 5 on page 908 of the book.

(c) Find the residue of $f(z)$ at $a = 5$ from your result in part (c).

(d) Compute the residue of $f(z)$ at $a = 5$ by using the formula

$$\operatorname{Res} f(a) = \frac{1}{(N-1)!} \lim_{z \to a} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \left[(z-a)^N f(z) \right] \ .$$

(e) Use the Residue Theorem to find $\displaystyle\oint_C \frac{\mathrm{e}^{3z}}{(z-5)^4}\, \mathrm{d}z$, where C is the contour $|z-3-\mathrm{i}| = 4$, traversed in positive direction. Draw the contour in $\mathbb{C}$.

(f) Compute the value of the integral from part (e) by using the generalized Cauchy integral formula (equation (7) on page 897 of the book).

Problem 8. Apply the Residue Theorem to evaluate

$$\oint_C \frac{1 - \cos z}{z^4}\, \mathrm{d}z \ ,$$

where C is the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$, traversed in positive direction.

Hint: First check carefully the order of the pole at 0. (It is *not* equal to 4.)

Problem 9.

(a) Show that $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{3 + \cos\theta} = \frac{\pi}{\sqrt{2}}$.

(b) Compute the value of the integral $\displaystyle\int_0^{\pi} \frac{\mathrm{d}\theta}{3 + \cos\theta}$.

Hint: Use the method described on page 930 of the book; your result from part (a) will be useful.