

Problem 1. Show that $\int_0^{2\pi} \frac{d\theta}{3 + \cos \theta} = \frac{\pi}{\sqrt{2}}$.

Problem 2. Show that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(1 + x^2)^2} = \frac{\pi}{2}$ by using the contour in Figure 19.5 on page 931.

Problem 3. Show that $\int_{-\infty}^{\infty} \frac{\cos(\pi x) dx}{x^2 - 2x + 2} = -\pi e^{-\pi}$ by using the contour in Figure 19.5.

Problem 4. The lines in \mathbb{C} defined by the equations

$$Z_1(t) = 1 + 5t + t^3 i, \quad Z_2(t) = e^t + i \sin(t\sqrt{3})$$

(where $t \in \mathbb{R}$) intersect at $t = 0$ at the point 1.

(a) Find $Z_1'(0)$ and $Z_2'(0)$.

(b) Find the angle between the tangents to $Z_1(t)$ and $Z_2(t)$ at their intersection for $t = 0$.

Hint: Write the complex numbers $Z_1'(0)$ and $Z_2'(0)$ in polar coordinates.

(c) Now think of the two curves as curves in \mathbb{R}^2 , namely, $\mathbf{r}_1(t) = (1 + 5t)\mathbf{i} + t^3\mathbf{j}$ and $\mathbf{r}_2(t) = e^t\mathbf{i} + \sin(t\sqrt{3})\mathbf{j}$, where \mathbf{i} and \mathbf{j} are the unit vectors in positive x - and y -directions, respectively. Find $\mathbf{r}_1'(0)$ and $\mathbf{r}_2'(0)$.

(d) Find $\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0)$ and use this to find the angle between the curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the point of their intersection for $t = 0$.

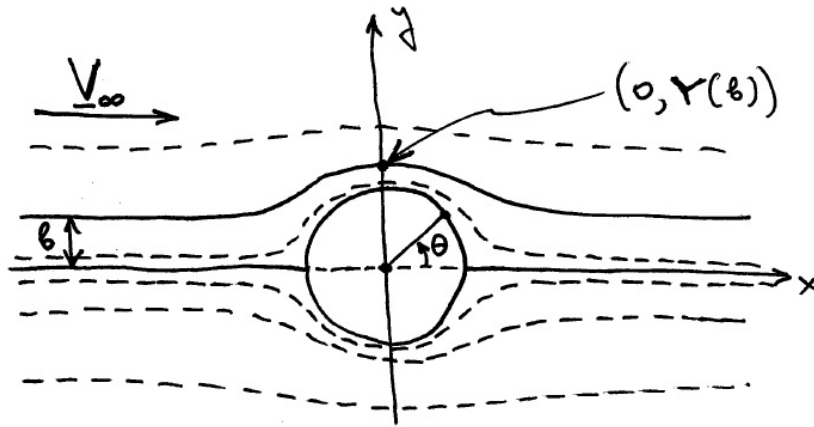
Problem 5. In this problem you will write down find the solution of the steady irrotational flow of incompressible inviscid fluid around a stationary disk of radius R ; the disk is impenetrable for the fluid. For convenience, we take the center of the disk to be placed at the origin of the coordinate system. Assume that far from the disk, the fluid is moving at a constant velocity \mathbf{V}_∞ in positive x -direction. In three dimensions, this situation corresponds to fluid flowing around an infinite cylinder of radius R whose axis coincides with the z -axis.

Consider the *complex potential*,

$$\Omega(z) = V_0 \left(z + \frac{R^2}{z} \right),$$

which is analytic in the exterior of the disk of radius R .

(a) Before starting with the complete problem, write $\Omega(z)$ very far from the disk, i.e., when $|z| \gg R$ (so that one of the terms in $\Omega(z)$ can be neglected). Let $\Omega_\infty(x + iy) =$



$\phi_\infty(x, y) + i\psi_\infty(x, y)$ be the new, simplified function at very large $|z|$. Find the velocity \mathbf{V}_∞ at very large $|z|$ by using the fact that $\mathbf{V}_\infty = \nabla\phi_\infty$. What is the speed $|\mathbf{V}_\infty|$ of the fluid very far from the disk? Discuss the physical meaning of the constant V_0 . What are the units for V_0 ?

- (b) Now we will study the full function $\Omega(z)$. Find the real and imaginary parts of $\Omega(z)$ in Cartesian coordinates, i.e., write $\Omega(z)$ in the form

$$\Omega(x + iy) = \phi(x, y) + i\psi(x, y) ,$$

where ϕ and ψ are real-valued functions of two variables.

- (c) Treat $\phi(x, y)$ as a velocity potential (see page 977), and find the velocity $\mathbf{V}(x, y) = \nabla\phi(x, y)$ of the fluid.
- (d) Write down the equation of the streamlines of the fluid $\psi(x, y) = C$ (where C is a constant).
- (e) Consider the equation $\psi(x, y) = 0$ (where the function ψ is the one coming from the particular choice of the function $\Omega(z)$ above). Sketch the curve(s) it describes, i.e., find the set of points $(x, y) \in \mathbb{R}^2$ that satisfy this equation. Discuss the physical meaning of your finding.

Hint: The answer of this question will reveal the reason of taking $\Omega(z)$ to have the particular form written above. Solving the equation $\psi(x, y) = 0$ is very easy!

- (f) Now consider the streamline going through a point of the form (x, b) , where b is a constant, and x is some very large negative number; this streamline is drawn with a solid line in the picture above. There is a streamline described by the equation $\psi(x, y) = C$ for some particular value of C that goes through the point (x, b) . Show that the value of C corresponding to this streamline is $C = V_0b$.

Hint: Show that $\lim_{x \rightarrow -\infty} \psi(x, b) = V_0b$.

- (g) Let $(-\infty, b)$ stand for the point (x, b) with $x \rightarrow -\infty$, like the point considered in the previous part of the problem. Find the value of y at which the streamline through the point $(-\infty, b)$ passes through the line $x = 0$. Denote this value by $Y(b)$ (see the picture above). You will obtain that $Y(b)$ satisfies the quadratic equation $\xi^2 - b\xi - R^2 = 0$. This equation always has two real roots – you have to choose the “physical” one (i.e., the one that behaves in a physically reasonable way).

Find $\lim_{b \rightarrow 0} Y(b)$ as well of the behavior of $Y(b)$ as b becomes very large. Do your results look physically reasonable?

- (h) Let us finally use Bernoulli’s equation,

$$\frac{1}{2}\rho v^2 + p = \text{const}$$

to find the pressure in the fluid. One can show (but the calculations are tedious) that the speed of the fluid is

$$|\mathbf{V}(x, y)| = V_0 \frac{\sqrt{[(x - R)^2 + y^2][(x + R)^2 + y^2]}}{x^2 + y^2}.$$

Let the pressure “at infinity” be p_∞ . Find the pressure of the fluid on the x -axis (more precisely, on the x -axis without the interval $(-R, R)$).

Find the pressure very close to the “surface” of the disk. To this end, let $z = Re^{i\theta} = R \cos \theta + iR \sin \theta$ be a point on the surface of the disk (i.e., set $x^2 + y^2 = R^2$ in the expressions above), and find the pressure on the surface of the disk as a function of θ .

In fact, from elementary physical considerations (and using Bernoulli’s equation), you can compute the pressures at the so-called “stagnation points” $(-R, 0)$ and $(R, 0)$ – compare the results for the pressure at these points you obtained above with these pressures obtained just from physical reasoning. (By definition, a “stagnation point” is a point at which the speed of a fluid is zero.)

Problem 6. The function

$$\phi(x, y) = x^2 - y^2 + 6xy$$

satisfies Laplace’s equation, $\Delta\phi(x, y) = 0$.

Using Cauchy-Riemann equations, find a function $\psi(x, y)$ such that

$$\Omega(z) = \phi(x, y) + i\psi(x, y)$$

for some analytic function $\Omega(z)$ of the complex variable $z = x + iy$.

Write the function $\Omega(z)$ explicitly.

Hint: Think about the following facts: $z^2 = x^2 - y^2 + 2xyi$, $iz^2 = -2xy + i(x^2 - y^2)$.