MATH 5763 Homework 5 Due Thu, 2/21/2019

Problem 1. Let X and Y be independent Poisson random variables with respective parameters λ and μ, i.e., $p_X(k) = \mathrm{e}^{-\lambda}\,\frac{\lambda^k}{k!}$, $p_Y(k) = \mathrm{e}^{-\mu}\,\frac{\mu^k}{k!}$, $k = 0, 1, 2, \ldots$. Show that:

(a) $X + Y$ is Poisson with parameter $\lambda + \mu$;

(b) the conditional distribution of X given that $X + Y = n$ is $\mathrm{Bin}(n, \frac{\lambda}{\lambda+\mu})$.

Hint: (b) Note that $\{X = k\} \cap \{X+Y = n\} = \{X = k\} \cap \{Y = n-k\}$, so that, by the independence of X and Y, $\mathbb{P}\left(\{X = k\} \cap \{X + Y = n\}\right) = \mathbb{P}\left(\{X = k\} \cap \{Y = n - k\}\right) = \mathbb{P}(X = k)\,\mathbb{P}(Y = n-k)$.

Problem 2. Let W_t and M_t be the number of women, respectively men, entering a big store in the time interval $[0, t]$. Assume that $W = \{W_t\}_{t\geq 0}$ and $M = \{M_t\}_{t\geq 0}$ are independent Poisson processes with intensities ω and μ, respectively.

(a) Prove that the total number of customers, $N_t := W_t + M_t$, entering the store forms a Poisson process with intensity $\omega + \mu$. Explain briefly why your result is obvious.

Hint: The solution follows easily from Problem 1.

(b) Find the conditional probability $\mathbb{P}(M_t = m \,|\, N_t = n)$, for $0 \leq m \leq n$, and show that M_t can be considered as a binomial random variable with one random and one deterministic parameter, namely $M_t \sim \mathrm{Bin}(N_t, \frac{\mu}{\omega+\mu})$. Why is this result obvious? Show that the conditional expectation $\mathbb{E}[M_t \,|\, N_t]$ is the random variable $\mathbb{E}[M_t \,|\, N_t] = \frac{\mu}{\omega+\mu}\, N_t$. (To answer the last question, you can use what you know about binomial random variables.)

(c) What is the probability that the first woman arrives at the store before the first man?

Hint: Here is a useful fact: if the random variables $E_1 \sim \mathrm{Exp}(\alpha_1)$ and $E_2 \sim \mathrm{Exp}(\alpha_2)$ are independent, then $\mathbb{P}(E_1 < E_2) = \dfrac{\alpha_1}{\alpha_1 + \alpha_2}$. You do not need to prove this fact in your homework, but I expect that you know how to prove it (see Proposition 3.3.4 on page 113 of Lefebvre's book).

[**Food for Thought only**] Could you have guessed the answer to the question in part (c) in the three limiting cases

- $\omega \to 0$ while μ is kept equal to a positive constant,
- $\mu \to 0$ while ω is kept equal to a positive constant,
- $\omega = \mu$?

Check that your general result obtained above behaves as expected in these three limits.

(d) Let H denote the arrival time of the first customer (it does not matter whether it is a woman or a man). What are the probability density function and the cumulative distribution function of the random variable H? (You can answer this question without any calculations, by using your answers to the previous parts of this problem.)

(e) What is the probability that during the first three hours (i.e., in the interval $[0,3]$), a total of exactly four customers have arrived at the store?

(f) Given that exactly four customers have arrived during the first three hours, what is the probability that all four of them were men?

[Food for Thought only] Could you have guessed the answer to this question in the three limiting cases from the "Food for Thought" part of question (c)? Does your general answer match your intuition in these limiting cases?

(g) Let T_1 denote the time of arrival of the first man at the store. Then W_{T_1} is the number of women that have arrived at the store by the time of the first man's arrival. Show that the probability distribution of the random variable W_{T_1} is

$$\mathbb{P}(W_{T_1} = k) = \frac{\omega^k \mu}{(\omega + \mu)^{k+1}} \qquad \text{for } k \in \mathbb{Z}_+ \ .$$

Hint: One can solve this problem in a couple of simple ways, or in a more complicated way. If you choose the complicated way, you may need to use that $\displaystyle\int_0^\infty x^k \, \mathrm{e}^{-x} \, \mathrm{d}x = \Gamma(k+1) = k!$ for $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

(h) **[Food for Thought only]** Consider your result in part (g) in the limiting case $\mu \to 0$ (while ω is kept at a fixed strictly positive value). Does your result seem strange? How do you explain this "paradox"?

Consider the other two limiting cases as in part (c) – do your results agree with your intuition?

Problem 3. The *probability generating function* (p.g.f.) of a random variable J taking only values in $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$ is defined as

$$G_J(\xi) := \mathbb{E}[\xi^J] = \sum_{j=0}^{\infty} p_j \xi^j \ , \qquad \text{where } \ p_j = \mathbb{P}(J = j) \ .$$

provided the right-hand side exists. Prove the following properties of the p.g.f.'s:

(a) $G_J(1) = 1$;

(b) $G'_J(1) = \mathbb{E}[J]$;

(c) $G''_J(1) = \mathbb{E}[J^2] - \mathbb{E}[J]$;

(d) $\mathrm{Var}\, J = G''_J(1) + G'_J(1) - [G'_J(1)]^2$;

(e) if $J_1, J_2, \ldots, J_r$ are i.i.d. random variables taking values in $\mathbb{Z}_+$, and $K = J_1 + \cdots + J_r$, then

$$G_K(x) = [G_J(x)]^r \ .$$

Please write explicitly where you use each of the assumptions.

Problem 4. A *death process* is a random process that describes the number of people in a society where the only reason for changing the number of people is dying (nobody is born, there is no immigration, etc.). We say that the random process $X = \{X_t : t \geq 0\}$ is a death process with parameter μ if each person dies independently of every other person, and the probability that each person dies in one unit of time is μ (we assume that the units of time we use are much shorter than the average lifetime of the people). Clearly, the probability of a death of a person in one unit of time in a population of i people is $i\mu$ (again, we assume that the unit of time is "short").

Here is the precise mathematical definition of a death process with parameter μ:

- the state space of the process is $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$;
- the process is non-increasing, i.e., if $s < t$, then $X_s \geq X_t$;
- if h is a very small positive number, then, for $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$\mathbb{P}(X_{t+h} = j | X_t = i) = \begin{cases} i\mu h + o(h) & \text{if } j = i - 1 \ , \\ 1 - i\mu h + o(h) & \text{if } j = i \ , \\ o(h) & \text{if } j > i \text{ or } j \leq i - 2 \ , \end{cases}$$

  and

$$\mathbb{P}(X_{t+h} = 0 | X_t = i) = \begin{cases} \mu h + o(h) & \text{if } i = 1 \ , \\ 1 & \text{if } i = 0 \ , \\ o(h) & \text{if } i > 1 \ . \end{cases}$$

  Here $o(h)$ is a function satisfying $\lim_{h \to 0} \frac{o(h)}{h} = 0$;
- for $s < t$, the difference $X_t - X_s$ (equal to the number of deaths in the interval $(s, t]$) does not depend on what has happened in the time interval $(0, s]$.

In this problem you will analyze some aspects of this process.

(a) Let $p_i(t) = \mathbb{P}(X_t = i)$. Condition on X_t to derive the equations

$$\begin{aligned} p_0(t + h) &= \mu h \, p_1(t) + p_0(t) + o(h) \ , \\ p_j(t + h) &= (j + 1)\mu h \, p_{j+1}(t) + (1 - j\mu h) \, p_j(t) + o(h) \ , \qquad j \in \mathbb{N} \ . \end{aligned}$$

(b) Subtract $p_j(t)$ from the jth equation from (a), divide through by h, and take the limit $h \to 0$, to obtain the system

$$\begin{aligned} p_0'(t) &= \mu \, p_1(t) \ , \\ p_j'(t) &= (j + 1)\mu \, p_{j+1}(t) - j\mu \, p_j(t) \ , \qquad j \in \mathbb{N} \ . \end{aligned}$$

Let the initial condition be $X_0 = I$, where I is a random variable taking values in $\mathbb{Z}_+$.

(c) Define the *generating function*

$$\Delta(\xi,t) := \sum_{j=0}^{\infty} p_j(t)\,\xi^j$$

and show that

$$\frac{\partial\Delta}{\partial\xi} = \sum_{j=0}^{\infty} j\,p_j(t)\,\xi^{j-1} = \sum_{j=1}^{\infty} j\,p_j(t)\,\xi^{j-1}\ , \qquad \frac{\partial\Delta}{\partial t} = \sum_{j=0}^{\infty} p_j'(t)\,\xi^j\ .$$

(d) Use the differential equations from part (b) to show that $\Delta(\xi,t)$ of the death process satisfies the partial differential equation

$$\frac{\partial\Delta}{\partial t} = \mu(1-\xi)\frac{\partial\Delta}{\partial\xi}\ ,$$

and the initial condition $\Delta(\xi,0) = \xi^I$ (where $I = X_0$ is the initial population).

Hint: Multiply the differential equation for $p_j'(t)$ by ξ^j and add all the equations.

(e) How can the probabilities $p_j(t) = \mathbb{P}(X_t = j)$ be expressed in terms of ξ-derivatives of $\Delta(\xi,t)$ evaluated at $\xi = 0$? Use this to find the explicit expressions for $\mathbb{P}(X_t = 0)$ and $\mathbb{P}(X_t = 1)$, using that the solution of the initial-value problem for the generating function posed in part (d) is

$$\Delta(\xi,t) = \left[1 + (\xi-1)\,\mathrm{e}^{-\mu t}\right]^I$$

(there is no need to derive this expression).

(f) Show that

$$\frac{\partial\Delta}{\partial\xi}(1,t) = \mathbb{E}[X_t]\ ,$$

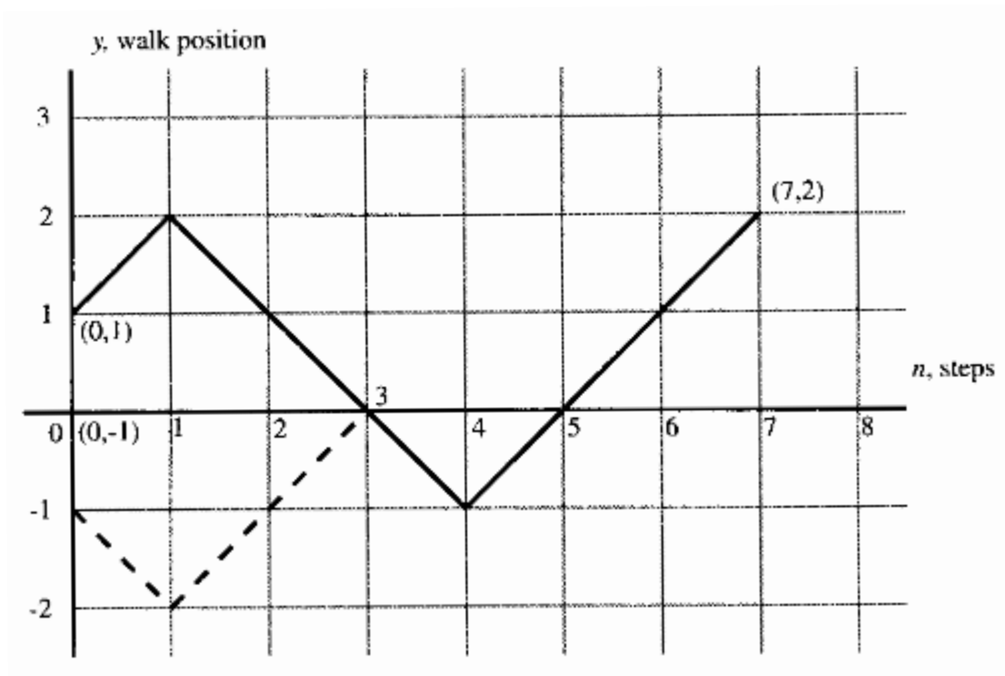
and use this fact to find $\mathbb{E}[X_t]$ for the death process.

(g) Using the same idea as in part (f), express the variance of X_t in terms of derivatives of its generating function evaluated at $\xi = 1$. Use the explicit expression for $\Delta(\xi,t)$ given in (e) to find $\operatorname{Var} X_t$ for the death process.

(h) Radioactive decay is an example of a death process, if we think of a nucleus of the radioactive isotope as "alive" before it decays, and "dead" after that. The "half-life", $T_{1/2}$, of a radioactive isotope is defined as the time after which only half of the initial number of nuclei of this isotope are "alive". How is $T_{1/2}$ related to μ? Justify your claim.

Food for Thought Problem 1. Consider an irreducible positive recurrent (discrete-time discrete-state space) Markov chain $\{X_n\}_{n=0}^{\infty}$, and assume that the initial state $X_0 = i$. Let $N_n(i)$ be the number of visits to state i in the first n trials, and $T_m(i)$ denote the number of trials until the mth visit to state i. Justify the relationship $\mathbb{P}(T_m(i) \geq n) = \mathbb{P}(N_n(i) \leq m)$. (Just give a convincing explanation in a couple of sentences.)

Food for Thought Problem 2. A random walk can be represented as a connected graph between coordinates (n, y), where the ordinate y is the position of the walk, and the abscissa n represents the number of steps. A walk of 7 steps which joins $(0, 1)$ and $(7, 2)$ is shown in the figure below. Suppose that a random walk starts at $(0, y_1)$ and finishes at (n, y_2), where $y_1 > 0$, $y_2 > 0$, and



$n + y_2 - y_1$ is an even number. Suppose also that the walk first visits the origin (i.e., position $y = 0$) at time $n = n_1$. Reflect that part of the path for which $n \le n_1$ in the n-axis (see the figure), and use a reflection argument to show that the number of paths from $(0, y_1)$ to (n, y_2) which touch or cross the n-axis is equal to the number of *all* pats from $(0, -y_1)$ to (n, y_2). This is known as the *reflection principle.*