MATH 5483 Homework 5 Due Thu, Feb 21

Problem 1. Let $\{g_n\}$ is any system of L^2 functions. Prove that $f \in \overline{\operatorname{span}}\{g_n\}$ if and only if there is a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that $f_k \in \operatorname{span}\{g_n\}$ and $\lim_{k\to\infty} \|f - f_k\|_2 = 0$.

Hint: For the "only if" direction, chose $f_k \in \operatorname{span}\{g_n\}$ such that $\|f - f_k\|_2 < \frac{1}{k}$.

Problem 2. Let the function ϕ satisfy the *two-scale dilation equation*

$$\phi(x) = \sum_{k\in\mathbb{Z}} h(k)\sqrt{2}\,\phi(2x-k) \qquad \text{in} \quad L^2(\mathbb{R})$$

for some ℓ^2 sequence of coefficients $\{h(k)\}_{k\in\mathbb{Z}}$.

(a) Prove that the Fourier transform $\widehat{\phi}$ of ϕ satisfies the equation

$$\widehat{\phi}(\gamma) = m_0\left(\frac{\gamma}{2}\right)\widehat{\phi}\left(\frac{\gamma}{2}\right) \ ,$$

where

$$m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-2\pi i k\gamma} \ .$$

(b) Find the coefficients $\{h(k)\}_{k\in\mathbb{Z}}$ and compute the function m_0 for the case of the Haar scaling function, $\phi = \chi_{[0,1)}$.

Problem 3. Reproduce the proof of part (b) of Lemma 7.16 from the book, filling out all details, and clearly pointing out what facts you use at each (in)equality.