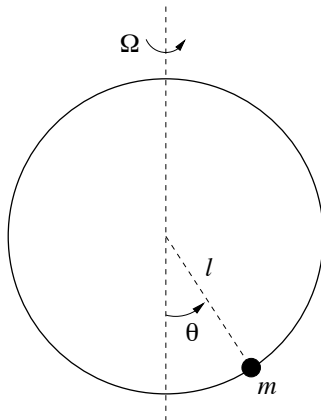


**Problem 1 (and only). [A bead on a rotating hoop]**

A bead of mass  $m$  can slide without friction on a circular hoop of radius  $\ell$  that rotates about a vertical diameter with constant angular speed  $\Omega$  as shown in the figure.



The equation of motion of the bead can be shown to be

$$m\ell \frac{d^2\theta}{dt^2} = m\ell \Omega^2 \cos \theta \sin \theta - mg \sin \theta , \quad (1)$$

where the angle  $\theta$  takes values in  $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ ; we think of the unit circle  $S^1$  as the interval  $(-\pi, \pi]$  with identified ends. By introducing the dimensionless time  $\tau := t\sqrt{\frac{g}{\ell}}$  and the non-negative dimensionless parameter  $\mu := \frac{\ell\Omega^2}{g} \geq 0$ , we can rewrite (1) as the system

$$\frac{d\theta}{d\tau} = \nu , \quad \frac{d\nu}{d\tau} = (\mu \cos \theta - 1) \sin \theta , \quad (2)$$

where  $\theta \in S^1$  and  $\nu \in \mathbb{R}$ , so that the pair  $(\theta, \nu)$  can be considered as an element of the infinite cylinder  $S^1 \times \mathbb{R}$ . The parameter  $\mu = \left( \frac{\Omega}{\sqrt{g/\ell}} \right)^2$  can be interpreted as follows:

$$\mu = \left( \frac{\text{angular velocity } \Omega \text{ of the hoop's rotation}}{\text{frequency } \sqrt{g/\ell} \text{ of small oscillations of the bead when the hoop is not rotating}} \right)^2 .$$

In this problem you will analyze the bifurcations in the system (2).

- (a) Find all fixed points (i.e., equilibrium solutions) of the system (2). Show that, for  $\mu \leq 1$  there are two equilibria (i.e., fixed points), while for  $\mu > 1$  there are four equilibria.

- (b) Linearize (2) around the fixed point  $(\pi, 0)$ . What kind of fixed point is it? Is it hyperbolic?

*Hint:* If (2) is written as  $\frac{d\mathbf{x}}{d\tau} = \mathbf{f}(\mathbf{x})$  with  $\mathbf{x} = \begin{pmatrix} \theta \\ \nu \end{pmatrix}$ , then

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \mu(\cos^2 \theta - \sin^2 \theta) - \cos \theta & 0 \end{pmatrix}.$$

- (c) In the case  $\boxed{\mu < 1}$ , linearize (2) around the fixed point  $(0, 0)$ , and show that  $(0, 0)$  is a center (hence, non-hyperbolic). Find and sketch the period  $T$  of the motion around this fixed point as a function of the parameter  $\mu$ .

*Hint:* Recall that if the eigenvalues  $\lambda_{1,2}$  of a matrix with real entries are not real, then they must be complex conjugates:  $\lambda_1 = \alpha + i\beta$ ,  $\lambda_2 = \alpha - i\beta$ . If the fixed point is a center (i.e.,  $\alpha = 0$ ), then the angular frequency of the small periodic motions around the fixed point is  $\beta$ , so that the period of these motions is  $\frac{2\pi}{|\beta|}$ .

- (d) In the case  $\boxed{\mu > 1}$ , linearize (2) around the fixed point  $(0, 0)$ . What kind of fixed point is  $(0, 0)$  in this case? Is it hyperbolic? Find its eigenvalues and eigenvectors.
- (e) In the case  $\boxed{\mu > 1}$ , linearize (2) around the fixed point  $(\arccos \frac{1}{\mu}, 0)$  and show that it is a center. Find the period  $T$  of the motion around this fixed point as a function of the parameter  $\mu$ , and sketch  $T(\mu)$ .
- (f) Use your results from (d) and (e) to sketch the phase portrait of the system in the case  $\boxed{\mu > 1}$ .

*Remark:* The behavior of the system around the fourth fixed point,  $(-\arccos \frac{1}{\mu}, 0)$  is the same as around  $(\arccos \frac{1}{\mu}, 0)$ .

- (g) Sketch the position of the four equilibria as functions of  $\mu$  (use solid line for the stable equilibria and dashed line for the unstable ones).
- (h) Find the positions of the four equilibria in the limit  $\mu \rightarrow \infty$ . What is the physical explanation of your result (in particular, in the limit  $\mu \rightarrow \infty$ )?
- (i) What is the physical explanation of the bifurcation occurring at  $\mu = 1$ ?
- (j) In Mathematica, execute the commands

```
StreamPlot[{y, (0.7*Cos[x] - 1)*Sin[x]}, {x, -Pi, Pi}, {y, -3, 3}]
```

and

```
StreamPlot[{y, (1.3*Cos[x] - 1)*Sin[x]}, {x, -Pi, Pi}, {y, -3, 3}]
```

Attach your printout and explain in several sentences what you observe and how it is related to your computations above.