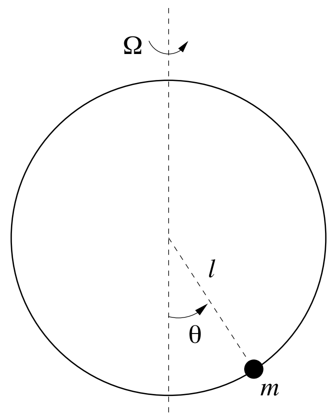
MATH 4513 Homework 5 Due Wed, Oct 21, 2015

Problem 1 (and only). [A bead on a rotating hoop]

A bead of mass m can slide without friction on a circular hoop of radius ℓ that rotates about a vertical diameter with constant angular speed Ω as shown in the figure.



The equation of motion of the bead can be shown to be

$$m\ell\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = m\ell\,\Omega^2\cos\theta\sin\theta - mg\sin\theta\ , \tag{1}$$

where the angle θ takes values in $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$; we think of the unit circle S^1 as the interval $(-\pi,\pi]$ with identified ends. By introducing the dimensionless time $\tau := t\sqrt{\frac{g}{\ell}}$ and the non-negative dimensionless parameter $\mu := \frac{\ell\Omega^2}{g} \geq 0$, we can rewrite (1) as the system

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \nu\ , \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\tau} = (\mu\cos\theta - 1)\sin\theta\ , \tag{2}$$

where $\theta \in S^1$ and $\nu \in \mathbb{R}$, so that the pair (θ,ν) can be considered as an element of the infinite cylinder $S^1\times\mathbb{R}$. The parameter $\mu = \left(\frac{\Omega}{\sqrt{g/\ell}}\right)^2$ can be interpreted as follows:

$$\mu = \left(\frac{\text{angular velocity } \Omega \text{ of the hoop's rotation}}{\text{frequency } \sqrt{g/\ell} \text{ of small oscillations of the bead when the hoop is not rotating}}\right)^2 .$$

In this problem you will analyze the bifurcations in the system (2).

(a) Find all fixed points (i.e., equilibrium solutions) of the system (2). Show that, for $\mu \leq 1$ there are two equilibria (i.e., fixed points), while for $\mu > 1$ there are four equilibria.

(b) Linearize (2) around the fixed point $(\pi, 0)$. What kind of fixed point is it? Is it hyperbolic?

Hint: If (2) is written as $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} = \begin{pmatrix} \theta \\ \nu \end{pmatrix}$, then

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ \mu(\cos^2\theta - \sin^2\theta) - \cos\theta & 0 \end{pmatrix}.$$

(c) In the case $\boxed{\mu < 1}$, linearize (2) around the fixed point $(0, 0)$, and show that $(0, 0)$ is a center (hence, non-hyperbolic). Find and sketch the period T of the motion around this fixed point as a function of the parameter μ.

Hint: Recall that if the eigenvalues $\lambda_{1,2}$ of a matrix with real entries are not real, then they must be complex conjugates: $\lambda_1 = \alpha + \mathrm{i}\beta$, $\lambda_2 = \alpha - \mathrm{i}\beta$. If the fixed point is a center (i.e., $\alpha = 0$), then the angular frequency of the small periodic motions around the fixed point is β, so that the period of these motions is $\dfrac{2\pi}{|\beta|}$.

(d) In the case $\boxed{\mu > 1}$, linearize (2) around the fixed point $(0, 0)$. What kind of fixed point is $(0, 0)$ in this case? Is it hyperbolic? Find its eigenvalues and eigenvectors.

(e) In the case $\boxed{\mu > 1}$, linearize (2) around the fixed point $(\arccos\frac{1}{\mu}, 0)$ and show that it is a center. Find the period T of the motion around this fixed point as a function of the parameter μ, and sketch $T(\mu)$.

(f) Use your results from (d) and (e) to sketch the phase portrait of the system in the case $\boxed{\mu > 1}$.

Remark: The behavior of the system around the fourth fixed point, $(-\arccos\frac{1}{\mu}, 0)$ is the same as around $(\arccos\frac{1}{\mu}, 0)$.

(g) Sketch the position of the four equilibria as functions of μ (use solid line for the stable equilibria and dashed line for the unstable ones).

(h) Find the positions of the four equilibria in the limit $\mu \to \infty$. What is the physical explanation of your result (in particular, in the limit $\mu \to \infty$)?

(i) What is the physical explanation of the bifurcation occurring at $\mu = 1$?

(j) In Mathematica, execute the commands

```
StreamPlot[{y, (0.7*Cos[x] - 1)*Sin[x]}, {x, -Pi, Pi}, {y, -3, 3}]
```

and

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StreamPlot[{y, (1.3*Cos[x] - 1)*Sin[x]}, {x, -Pi, Pi}, {y, -3, 3}]
```

Attach your printout and explain in several sentences what you observe and how it is related to your computations above.