MATH 4193/5103 **Homework 5** **Due 4/18/17 (Tuesday)**

Problem 1. [Thinking simply, yet again]
The total resistance of two resistors, R_1 and R_2, in series is

$$R_{\text{in series}} = R_1 + R_2 \ ,$$

while their total resistance in parallel is given by

$$\frac{1}{R_{\text{in parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} \ .$$

Use these facts to find the total resistance between the points A and D (the leftmost points) of the infinite chain of resistors drawn in Figure 1. The resistance of each resistor is $R = 1\,\text{Ohm}$. You can find the total resistance very simply, if you think similarly to Problem 1 from Homework 4.

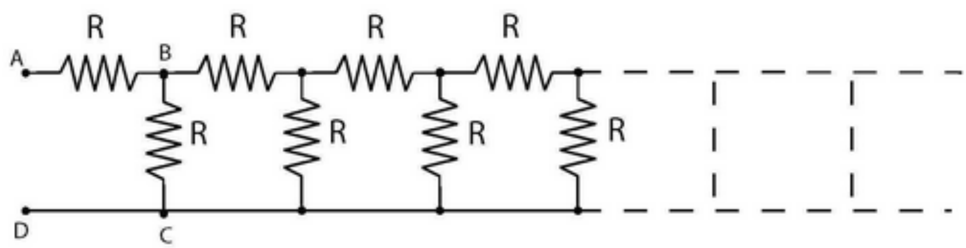

Figure 1: Infinite chain of resistors.

Problem 2. [Poincaré-Bendixson Theorem]
Consider the system

$$\dot{x} = x - y - x(x^2 + 5y^2) \ , \qquad \dot{y} = x + y - y(x^2 + y^2) \ . \tag{1}$$

(a) Classify the fixed point at the origin.

(b) Rewrite the system in polar coordinates, using that $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$.

Hint: The answer is $\dot{r} = r - r^3 - 4r^3 \cos^2\theta \sin^2\theta$, $\dot{\theta} = 1 + 4r^2 \cos\theta \sin^3\theta$, but I want to see your calculations.

(c) Prove that the maximum value of the function $\varphi(\theta) := (\cos\theta \sin\theta)^2$ is $\frac{1}{4}$ (if θ is allowed to take any value). The easiest way to answer this question is to use some *very* elementary trigonometry. What is the minimum value that the function $\varphi(\theta)$ takes?

(d) Determine the circle of maximum radius, r_1, centered at the origin such that all trajectories have a radially outward component on it.

(e) Determine the circle of minimum radius, r_2, centered at the origin such that all trajectories have a radially inward component on it.

(f) Prove that the system (1) has a limit cycle in the trapping region $r_1 \le r \le r_2$. Figure 2 shows the result of numerical integration of the system (1). If you are taking the class as MATH 4193, you may assume without proof that there are no fixed points of the system (1) in the trapping region (this follows directly from part (g) which is only for those taking the class as 5103).

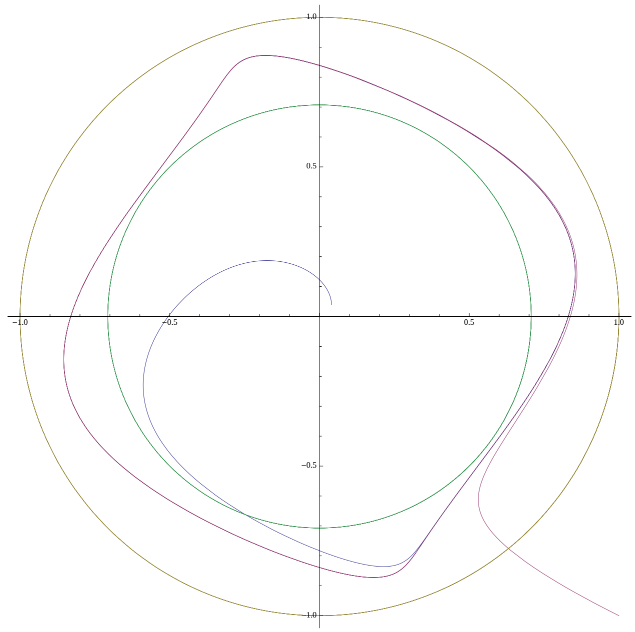

Figure 2: A limit cycle of the system (1), two phase trajectories, and the boundaries of the trapping region.

(g) **Only if you take the class as 5103!**

Show that there are no fixed points in the trapping region found in part (f).

Hint: One way to prove this is to exclude r from the system $\frac{\dot{r}}{r} = 0$, $\dot{\theta} = 0$, and then to show that the resulting equation for θ has no solution. To avoid long calculations, you may use a computer to plot some function of one variable.

Problem 3. [Nullclines and bifurcations]

In the article

P. Gray and S. Scott, Sustained oscillations and other exotic patterns of behavior in isothermal reactions, *Journal of Physical Chemistry*, Vol. 89 (1985), pp. 22–32 the authors consider a hypothetical isothermal autocatalytic reaction whose kinetics are given in dimensionless form by the two-parameter system

$$\begin{aligned} \dot{x} &= a(1-x) - xy^2 \ , \\ \dot{y} &= xy^2 - (a+b)y \ . \end{aligned} \tag{2}$$

Here $a > 0$ and $b > 0$ are positive parameters; the functions $x(t)$ and $y(t)$ may take any values in $\mathbb{R}$.

(a) Show that the equations of the nullclines can be written as follows:

$$\begin{aligned} (\dot{x}=0)\text{-nullcline}: &\quad x = \varphi(y) := \frac{a}{a+y^2} \ , \\ (\dot{y}=0)\text{-nullcline}: &\quad x = \psi(y) := \frac{a+b}{y} \quad \text{or} \quad y \equiv 0 \ . \end{aligned} \tag{3}$$

(b) Prove that at $b = -a \pm \frac{1}{2}\sqrt{a}$, the nullclines $\{x = \varphi(y)\}$ and $\{x = \psi(y)\}$ become tangent. Show that the coordinates $(x^*_\pm, y^*_\pm)$ of the points where the tangencies occur are $(\frac{1}{2}, \pm\sqrt{a})$. Note that the same sign (either $+$ or $-$) should be used in the expression for b and in the expression for $(x^*_\pm, y^*_\pm)$.

Remark: It is clear that the nullclines $\{x = \varphi(y)\}$ and $\{y \equiv 0\}$ can never be tangent, so you do not need to consider this.

(c) What is the significance of what you found in part (b)? Discuss its meaning from point of view of the bifurcations occurring in the systems, naming specifically the type of bifurcations that occurs.

(d) Take $a = 1$ and consider the $+$ sign in all expressions from part (b). We know from part (b) that when $b = -1 + \frac{1}{2}\sqrt{1} = -\frac{1}{2}$, the nullclines will be tangent at the point $(x^*_+, y^*_+) = (\frac{1}{2}, 1)$. Let us consider what happens if b is slightly off, i.e., take $b = -\frac{1}{2} + \xi$, where ξ is a very small number (positive or negative). For these values of a and b, the equations (3) of the nullclines become $x = \frac{1}{1+y^2}$, $x = \frac{\frac{1}{2}+\xi}{y}$. Show that from this system we can obtain the following quadratic equation for y:

$$y^2 - (\tfrac{1}{2} + \xi)^{-1}\, y + 1 = 0 \ .$$

Since ξ is very small, we have $\frac{1}{\frac{1}{2}+\xi} = \frac{2}{1-(-2\xi)} = 2\,[1 + (-2\xi) + (-2\xi)^2 + \cdots] \approx 2(1-2\xi)$, so in this approximation the quadratic equation for y becomes

$$y^2 - 2(1-2\xi)\, y + 1 = 0 \ .$$

Show that, neglecting all high-order in ξ terms, we can write the solutions of this quadratic equation as $y_{1,2} = 1 - 2\xi \pm 2\sqrt{-\xi}$. Discuss why there are no solutions for $\xi > 0$ while there are two solutions for $\xi < 0$.

(e) Leaving only the term of lowest order with respect to ξ, we can write $y_{1,2} \approx 1 \pm 2\sqrt{-\xi}$. In the same approximation, find the corresponding values $x_{1,2}$. Please write your calculations in detail.