## MATH 4193/5103 Homework 5 Due Thu, May 2, 2013

**Problem 1. [Scaling near a homoclinic bifurcation]**

To find how the period of a closed orbit scales as a homoclinic bifurcation is approached, one can perform a simple estimate of the time it takes for a trajectory to pass near a saddle point – this time is the "bottleneck", i.e., it is much longer than the time it takes to traverse the rest of the closed orbit. Suppose that the system is given locally by

$$\dot{x} \approx \lambda_u x \ , \qquad \dot{y} \approx -\lambda_s y \ , \tag{1}$$

where $\lambda_u$ and $\lambda_s$ are positive constants. Clearly, $(0,0)$ is a fixed point of the system (1), and the unstable, $W^u_{(0,0)}$, and stable, $W^s_{(0,0)}$, manifolds of this fixed point are (approximately) the $x$-axis and the $y$-axis, respectively.

Let a trajectory pass through the point $(\mu, 1)$, where $0 < \mu \ll 1$ is the distance from the stable manifold. How long does it take until the trajectory has escaped from the saddle point $(0,0)$, say, out to $x(t) = 1$?

**Problem 2. [Perturbed motion in a central force and quasiperiodicity]**

Consider a particle of negligible size moving in $\mathbb{R}^3$ that is being acted upon by a *central force*, i.e., a force that at any moment of time is directed towards a fixed point in $\mathbb{R}^3$; without loss of generality, let us assume that this point is the origin, $(0,0,0) \in \mathbb{R}^3$. An important physical example of such situation is the so-called *Kepler problem*, in which the force acting on the particle is inversely proportional to the square of the distance between the particle and the origin.

A very useful fact about central forces is that the *angular momentum* of the motion of the particle is conserved (i.e., constant with time). This implies that the motion of the particle in a central force always occurs in a fixed plane, so that one can forget about the third dimension, and consider the motion of the particle in $\mathbb{R}^2$.

The equations governing the motion of the particle can be written in polar coordinates in the form

$$\ddot{r} = -\frac{F(r)}{m} + \frac{L^2}{m^2 r^3} \ , \qquad \dot{\theta} = \frac{L}{mr^2} \ , \tag{2}$$

where $F(r)$ is the magnitude of the attracting force, $m$ is the mass of the particle, and $L = \text{const} > 0$ is the angular momentum (the unit for $L$ is $\frac{\mathrm{kg\,m}^2}{\mathrm{s}}$); as explained above, $L$ does not change with time. As usual, $r(t)$ and $\theta(t)$ stand for the polar coordinates of the particle at time $t$. A derivation of (2) can be found, e.g., in L.D. Landau, E.M. Lifshitz, *Mechanics*, 3rd ed., Butterworth-Heinemann, 1976, Sections 14 and 15.

In this problem you will consider the motion of a particle that occurs near a periodic trajectory. You will study two cases – Case A, in which the attracting force is gravity ($F(r) \sim r^{-2}$), and Case B, in which case the attracting force has a constant magnitude ($F(r) = k = \text{const}$).

($A_0$) When the motion of the particle is governed by Newton's law of gravity, the equations (2) have the form

$$\ddot{r} = -\frac{GM}{r^2} + \frac{L^2}{m^2 r^3} , \qquad \dot{\theta} = \frac{L}{mr^2} , \tag{3}$$

where $G$ denotes Newton's constant of gravitation (measured in $\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$), and $M$ is the mass of the large attracting body at the origin of the coordinate system.

($A_1$) Show that the system (3) has a solution $r = r_0$, $\dot{\theta} = \omega_\theta$, corresponding to uniform circular motion at a radius $r_0$ and angular frequency $\omega_\theta$ (hence, the period of the angular motion of the system is $\frac{2\pi}{\omega_\theta}$). Express $r_0$ and $\omega_\theta$ in terms of $G$, $M$, $m$, and $L$.

*Hint:* You will obtain that $r_0 = \frac{L^2}{GMm^2}$.

($A_2$) Now assume that the uniform circular motion of the particle is slightly perturbed in radial direction, i.e., that $r(t) = r_0 + u(t)$, where $r_0$ is the expression found in part ($A_1$) and $u(t)$ is a small perburbation, such that $|u(t)| \ll r_0$. Substitute $r(t) = r_0 + u(t)$ in the equation for $r(t)$ in (3) and obtain a second-order linear differential equation for $u(t)$, keeping only linear with respect to $\frac{u(t)}{r_0}$ terms in the right-hand side. You can use that, if $x$ is much smaller than 1, then

$$(1 + x)^\alpha \approx 1 + \alpha x \qquad \text{for any } \alpha \in \mathbb{R} .$$

You will obtain that $u(t)$ satisfies the harmonic oscillator equation with frequency $\omega_r = \frac{G^2 M^2 m^3}{L^3}$. Please write your calculations in detail.

($A_3$) What is the ratio of the frequencies $\omega_\theta$ (found in part ($A_1$)) and $\omega_r$ (found in part ($A_2$))?

*Remark:* It will turn out that the ratio $\frac{\omega_r}{\omega_\theta}$ is a rational number. If you think of the pair of periodic functions $u(t)$ and $\theta(t)$ as taking values in the two-dimensional torus, i.e., $(u(t), \theta(t)) \in \mathbb{T}^2$, then the fact that $\frac{\omega_r}{\omega_\theta}$ is a rational number implies that the trajectory $\{(u(t), \theta(t)) : t \in \mathbb{R}\}$ is a closed line in $\mathbb{T}^2$. The fact that the perturbed trajectory is again closed is, as it turns out, a quite exceptional property of the Newton's law of gravity.

($B_0$) Now assume that the force with which the particle is attracted to the origin is constant; in this case the motion of the particle is described by the equations

$$\ddot{r} = -\frac{k}{m} + \frac{L^2}{m^2 r^3} , \qquad \dot{\theta} = \frac{L}{mr^2} , \tag{4}$$

where $k$ is a positive constant (measured in $\frac{\mathrm{kg\,m}}{\mathrm{s}^2}$).

($B_1$) Show that the system (4) has a solution $r = r_0$, $\dot{\theta} = \omega_\theta$, corresponding to uniform circular motion at a radius $r_0$ and angular frequency $\omega_\theta$. Express $r_0$ and $\omega_\theta$ in terms of $k$, $m$, and $L$.

*Hint:* Clearly, the expressions you will obtain will differ from the ones obtained in part ($A_1$).

($B_2$) Repeat what you did in part ($A_2$) for the system (4) (using the expressions obtained in part ($B_1$)), to find the frequency $\omega_r$ of the oscillation of the perturbation $u(t)$ to the uniform circular motion.

($B_3$) What is the ratio of the frequencies $\omega_\theta$ (found in part ($B_1$)) and $\omega_r$ (found in part ($B_2$))?

*Remark:* In this case the ratio of the two frequencies will be an irrational number, which corresponds to a quasiperiodic motion on the two-dimensional torus, so that the trajectory $\{(u(t), \theta(t)) : t \in \mathbb{R}\}$ will fill $\mathbb{T}^2$ densely (and will never close).

**Problem 3. [A very simple Poincaré map]**

Consider the system

$$\dot{\theta} = 1 \ , \qquad \dot{y} = ay \tag{5}$$

on the cylinder $S^1 \times \mathbb{R}$, i.e., $\theta(t)$ belongs to the circle $S^1$, while $y(t) \in \mathbb{R}$. The number $a$ is an arbitrary real constant.

Define an appropriate Poincaré map and find a formula for it. Show that the system (5) has a periodic orbit. Classify the stability of this periodic orbit for all real values of $a$.

**Problem 4. [A more complicated Poincaré map]**

Consider the system

$$\dot{r} = r - r^2 \ , \qquad \dot{\theta} = 1 \ , \tag{6}$$

where $(r, \theta)$ are the polar coordinates in $\mathbb{R}^2$.

(a) Find the solution $(r(t), \theta(t))$ of (6), with initial conditions $(r(0), \theta(0)) = (r_0, \theta_0)$.

*Hint:* The following identity may be useful: $\dfrac{1}{r(1-r)} = \dfrac{1}{r} + \dfrac{1}{1-r}$.

(b) Let the surface of section, $\Sigma$, be the positive $x$-axis (i.e., the set of points with $\theta = 0$). Compute the Poincaré map from $\Sigma$ to itself.

(c) Show that the Poincaré map $P : \Sigma \to \Sigma$ obtained in part (b) has a unique fixed point.

(d) Classify the stability of the fixed point of $P$ found in part (c).

*Hint:* You may find useful the fact that $\dfrac{\mathrm{d}}{\mathrm{d}\xi} \dfrac{1}{1 + \mathrm{e}^{-2\pi}(\frac{1}{\xi} - 1)} = \dfrac{\mathrm{e}^{-2\pi}}{\xi^2 \big[1 + \mathrm{e}^{-2\pi}(\frac{1}{\xi} - 1)\big]^2}$.

(e) Interpret your results from parts (c) and (d) in terms of the existence and stability of a periodic orbit of the system (6).

(f) Find the Floquet multiplier for the periodic orbit.

*Hint:* The *Floquet multipliers* (or *characteristic multipliers*) of a periodic orbits are by definition of the eigenvalues of the matrix $D\vec{P}(\vec{\xi}^*)$ (where $\vec{\xi}^* \in \Sigma$ is the point where the periodic orbit intersects the surface of section $\Sigma$); see page 282 of Strogatz.

**Problem 5. [Variational system]** $\boxed{\textbf{Only if you take the class as 5103!}}$
Consider the autonomous nonlinear system

$$\dot{x} = -x \ , \qquad \dot{y} = x^2 + y \ . \tag{7}$$

(a) Solve the initial value problem (7) with arbitrary initial conditions, $(x(0), y(0)) = (x_0, y_0)$.

*Hint:* Although the system (7) is nonlinear, it can be solved easily – first find $x(t)$, and then plug this function in the second equation and solve it for $y(t)$. You should obtain that $x(t) = x_0\,\mathrm{e}^{-t}$, $y(t) = \left(y_0 + \frac{x_0^2}{3}\right)\mathrm{e}^{t} - \frac{x_0^2}{3}\,\mathrm{e}^{-2t}$.

(b) Let $\mathbf{\Phi}_t(\mathbf{x}_0)$ be the flow of the of the autonomous nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on $\mathbb{R}^n$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ (in other words, $\mathbf{\Phi}_t(\mathbf{x}_0)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}_t(\mathbf{x}_0) = \mathbf{f}(\mathbf{\Phi}_t(\mathbf{x}_0))$ and $\mathbf{\Phi}_0(\mathbf{x}_0) = \mathbf{x}_0$).

Suppose that we know the solution $\mathbf{\Phi}_t(\mathbf{x}_0)$ and want to find, at least approximately, the behavior of small perturbations of this solution. To this end, we can study the behavior of $\mathbf{\Phi}_t(\mathbf{x}_0 + \mathbf{u}_0)$ for very small $\mathbf{u}_0 \in \mathbb{R}^n$. We do not want to solve the nonlinear system (7) again, so instead we define the vector-valued functions

$$\mathbf{u}(t) := \mathbf{\Phi}_t(\mathbf{x}_0 + \mathbf{u}_0) - \mathbf{\Phi}_t(\mathbf{x}_0)$$

that give us the time evolution of the small perturbations to the original solution. Clearly, $\mathbf{u}(t)$ must satisfy

$$\begin{aligned}
\dot{\mathbf{u}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{\Phi}_t(\mathbf{x}_0 + \mathbf{u}_0) - \mathbf{\Phi}_t(\mathbf{x}_0)\right] \\
&= \mathbf{f}\left(\mathbf{\Phi}_t(\mathbf{x}_0 + \mathbf{u}_0)\right) - \mathbf{f}\left(\mathbf{\Phi}_t(\mathbf{x}_0)\right) \\
&= D\mathbf{f}\left(\mathbf{\Phi}_t(\mathbf{x}_0)\right)\left[\mathbf{\Phi}_t(\mathbf{x}_0 + \mathbf{u}_0) - \mathbf{\Phi}_t(\mathbf{x}_0)\right] + O\left(\left|\mathbf{\Phi}_t(\mathbf{x}_0 + \mathbf{u}_0) - \mathbf{\Phi}_t(\mathbf{x}_0)\right|^2\right) \\
&= D\mathbf{f}\left(\mathbf{\Phi}_t(\mathbf{x}_0)\right)\mathbf{u}(t) + o\left(|\mathbf{u}(t)|^2\right)
\end{aligned}$$

and

$$\mathbf{u}(0) = \mathbf{\Phi}_0(\mathbf{x}_0 + \mathbf{u}_0) - \mathbf{\Phi}_0(\mathbf{x}_0) = (\mathbf{x}_0 + \mathbf{u}_0) - \mathbf{x}_0 = \mathbf{u}_0 \ .$$

This motivates the definition of the *variational equation* for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ around the solution $\mathbf{\Phi}_t(\mathbf{x}_0)$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ as

$$\dot{\mathbf{u}}(t) = D\mathbf{f}\left(\mathbf{\Phi}_t(\mathbf{x}_0)\right)\mathbf{u}(t) \tag{8}$$

or, in components,

$$\dot{u}_i(t) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\left(\mathbf{\Phi}_t(\mathbf{x}_0)\right) u_j(t) \ .$$

The initial condition for $\mathbf{u}(t)$ is $\mathbf{u}(0) = \mathbf{u}_0$ (where $\mathbf{u}_0$ is arbitrary).

The advantage of solving (8) instead of the original system (7) is that (8) is *linear* (although, generally, nonautonomous).

Write down the variational equation of the system (7) around the solution found in part (a).

(c) Solve the variational equation that you wrote in part (b) for the particular choice $\mathbf{x}_0 = \mathbf{0}$, with arbitrary initial condition $\mathbf{u}_0$.

*Hint:* You should obtain that

$$u(t) = u_0\,\mathrm{e}^{-t}\,, \qquad v(t) = -\frac{2}{3}x_0u_0\,\mathrm{e}^{-2t} + \left(v_0 + \frac{2}{3}x_0u_0\right)\mathrm{e}^{t}\,.$$

(d) How does the solution $\mathbf{u}(t)$ found in part (c) behave as $t \to \infty$? Are there initial conditions $\mathbf{u}_0 \neq \mathbf{0}$ for which $\mathbf{u}(t)$ stays bounded as $t \to \infty$?